Azim Premji University

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At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 4, No.1
March 2015

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2}$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}+\cdots=\frac{1}{3}$$

PULLOUT
GEOMETRY - II

Featured in this issue of *At Right Angles* are several 'Proofs Without Words' or PWWs as they are fondly known. Displayed on the cover are three pictures that illustrate PWWs for three geometric identities (see http://memtropy.com/proofs-without-words/, https://math.stackexchange.com/questions/733754/visually-stunning-math-concepts-which-are-easy-to-explain and http://jeremykun.com/tag/proofs-without-words/page/2/):

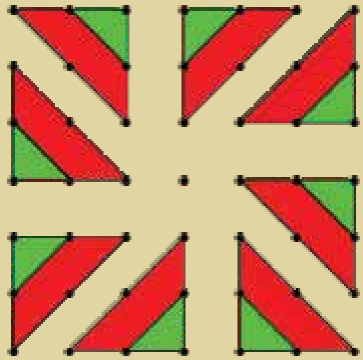
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1,$$

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{2},$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}.$$

We readily see the pattern in these identities. Remarkably, it is possible to design pictures that illustrate each of the relations.

Here is a picture which illustrates why "eight times a triangular number plus 1 is a perfect square". Here the 'triangular numbers' are the numbers 1, 3, 6, 10, 15, 21, ..., obtained by summing the positive integers from 1 to a prescribed upper limit. (See the article *Triangular Numbers* in the July 2012 issue of 'Resonance' for more identities of this kind.)



The theme is rich and pleasing, with immense scope for indulging one's artistic inclinations. Enjoy!

From The Editor's Desk . . .

"Every now and then one paints a picture that seems to have opened a door and serves as a stepping stone to other things." — **Pablo Picasso**

An apt saying for our lead feature article by CoMaC on *Proof Without* Words. Laced with thought-provoking examples and supplemented with a proof of Viviani's theorem by V.G. Tikekar, the article discusses the seemingly self-contradictory nomenclature of this sort of proof. A theme which is dwelt on by Punya Mishra and Gaurav Bhatnagar in their trademark quirky style in Part I of their two part series on *Paradoxes*. We also feature a new author this time: Ali Hussen writes on approximations to square roots and cube roots of numbers – his article weaves in algebra, geometry and arithmetic to explain his algorithm. And Shailesh Shirali explains in a complementary article exactly why this algorithm works!

In the Classroom features four articles, each of which can bring that extra element into the math teacher's repertoire. A. Ramachandran writes on the angle bisectors of a quadrilateral, while Shailesh Shirali illustrates the theme of *Generalization and Specialization* with some fascinating examples. The series on *How to Prove It* continues to offer more strategies. With this issue we begin a new series on *Low Floor High Ceiling* activities – we hope that you are sufficiently intrigued to read this article and try the activities described in it.

Thomas Lingefjard, in his article *Learning Math with a DGE system* addresses a pressing need of teachers using technology in the classroom: How does mathematical content knowledge and skill development improve with the use of educational software?

Problem Corner continues to provide much food for thought for our readers: we welcome your solutions to the featured problems. In the Review section, Mark Kleiner discusses Edward Frenkel's *Love and Math – the Heart of Hidden Reality.* We conclude with the pullout by Padmapriya Shirali, *Teaching of Geometry* (Part II).

Now for some advances made by *At Right Angles.* Our Pullout Section is now available in Kannada with a Hindi version soon to follow. These will soon be available online on http://teachersofindia.org/en/periodicals/at-right-angles.

We are also delighted to announce our FaceBook page 'AtRiUM' –At Right Angles Us and Math at https://www.facebook.com/groups/ 829467740417717/. It hopes to build a community of 'At Right Anglers' – readers who are excited about mathematics and connect with the larger discourse to grow as individuals and practitioners. While the magazine provides in-depth insights into the history and practice of mathematics, this space fosters sharing along these lines and encourages conversations about investigations, explorations, and innovative practices in the teaching and learning of the subject. Do explore and join the group. We would love to hear from you.

— **Sneha Titus**
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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an ‘academic’ and 'practitioner’ oriented magazine.

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Pullout

The art …

A Proof Without Words for Viviani's Theorem

…of mathematics

feature

Readers of this magazine may recall that in the December 2012 issue we dwelt on Viviani's theorem and considered different ways of proving it. In the process we studied a few extensions and generalizations of the theorem. In this article we return to this theorem and discuss a proof-without-words (PWW) for the result. See [3] for the original version of the proof.

V G TIKEKAR

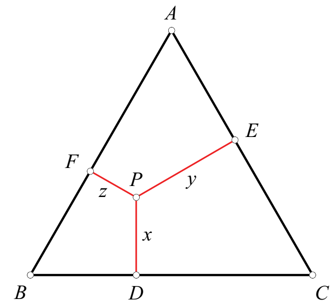


Figure 1. Viviani's theorem: $x + y + z =$ constant

Here is the statement of the theorem. Let $\triangle ABC$ be equilateral, and let P be any point in its interior (Figure 1). Then: *The sum of the distances from P to the sides of the triangle is a constant*. Thus, if perpendiculars PD, PE, PF are drawn from P to the sides BC, CA, AB, and their lengths are x, y, z, respectively, the sum $PD + PE + PF = x + y + z$ is the same for all positions of P.

Keywords: *Viviani, proof without words, equilateral triangle, translation, congruence, altitude*

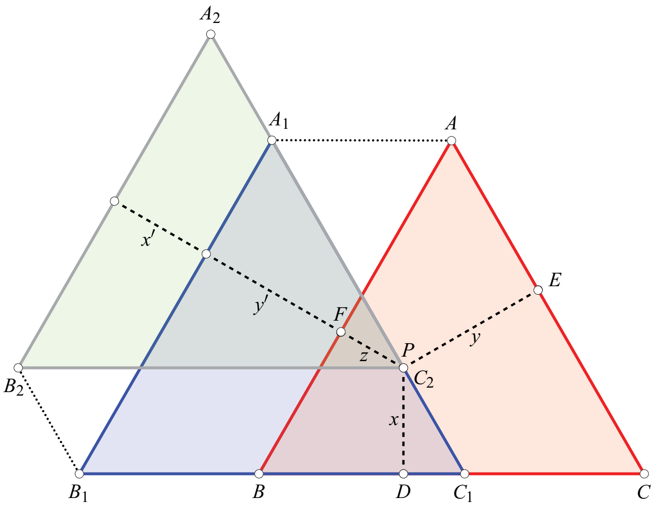


Figure 2. PWW for Viviani´s theorem. Note that $P = C_2$

In Figure 2 we see the proposed PWW. We start by translating $\triangle ABC$ along line BC to position $A_1B_1C_1$ so that point P lies on side A_1C_1. Then we translate $\triangle A_1B_1C_1$ along line A_1C_1 to position $A_2B_2C_2$ so that P coincides with vertex C_2. By construction, therefore, $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are congruent copies of $\triangle ABC$, and $A_2A_1C_1$ is parallel to AC, and B_2C_2 is parallel to B_1BC_1C. From P we drop a perpendicular to A_1B_1 and A_2B_2. Let the distance between the parallel lines AB and A_1B_1 be y', and let the distance between the parallel lines A_1B_1 and A_2B_2 be x'. (The symbols y' and x' are marked in the figure.)

Since parallelograms ABB_1A_1 and ACC_1A_1 are congruent to each other, the distances between corresponding pairs of opposite edges are equal; hence $y' = y$.

Similarly, since parallelograms $B_1C_1C_2B_2$ and $B_1A_1A_2B_2$ are congruent to each other, $x' = x$. Therefore $x + y + z = z + y' + x'$.

But $z + y' + x'$ equals the height of $\triangle A_2B_2C_2$, which is the same as the height of $\triangle ABC$. It follows that $x + y + z$ is equal to the height of $\triangle ABC$.

For another PWW of Viviani's theorem, see [1]. [Note from the editor: This issue of *At Right Angles* has a separate article on the subject of PWWs.]

References

[1] Kawasaki, K. *Proof Without Words: Viviani's Theorem*. https://www.maa.org/sites/default/files/3004415840629.pdf.bannered.pdf

[2] Bogomolny, A. *Viviani's Theorem: What is it?*. From "Interactive Mathematics Miscellany and Puzzles". http://www.cut-the-knot.org/Curriculum/Geometry/Viviani.shtml, Accessed 18 November 2014

[3] Nelsen, Roger B. *Proofs Without Words: Exercises in Visual Thinking*, page 15. MAA

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

feature

Is a picture worth ...

On Proofs Without Words

...a thousand words?

A "proof without words" sounds like a contradiction in terms! How can you prove something if you are not permitted the use of any words? In spite of the seeming absurdity of the idea, the notion of a proof without words — generally shortened to PWW — has acquired great popularity in mathematics in recent decades, and every now and then we come across new, elegant PWWs for old, familiar propositions. In this short article the seemingly contradictory nature of a PWW is discussed, and some examples of PWWs are presented.

C⊗MαC

Introductory remarks. In recent decades there has been much interest in "proofs without words" (PWWs for short) which, as the Math Wolfram source [6] compactly puts it, are proofs "...only based on visual elements, without any comments." PWWs today form a whole new genre of proofs. Two of the magazines published by the Mathematical Association of America (MAA) — *College Journal of Mathematics* and *Mathematics Magazine* — regularly publish original PWWs sent in by readers. Two PWW anthologies ([2] and [3]) have appeared in book form (they contain nothing but PWWs), largely culled from the magazines mentioned, and there are a few web pages too ([1], [4], [5]) which have nice collections of their own.

What exactly is a PWW? The Wikipedia source [5] has the following to say: "In mathematics, a proof without words is a proof of an identity or mathematical statement which can be

Keywords: *Visual proof, Pythagoras theorem, cosine rule, tan 15, triangular number, arithmetic mean, geometric mean, harmonic mean*

demonstrated as self-evident by a diagram without any accompanying explanatory text. Such proofs can be considered more elegant than more formal and mathematically rigorous proofs due to their self-evident nature. When the diagram demonstrates a particular case of a general statement, to be a proof, it must be generalizable."

But can there really be such a thing as a proof without words, or is it a self-contradictory notion? Consider what a mathematical proof is supposed to be: an argument written out in clear, understandable language, starting with a given set of propositions, justifying each step (generally done by referring to a proposition that has already been proved), and culminating in the proposition to be proved. Thus, every step is made *formal* and *explicit*.

At least that's the way it is supposed to be. In practice there are lots of statements which are not justified, on the ground that they are "obvious". Look through any proof and sooner or later you will meet these phrases: "it should be clear that …", "now obviously …", "it is quite obvious that …"; or phrases similar to these in meaning and intent. Perhaps that's the way it has to be; how can one possibly justify every single statement? The following fact is noteworthy: when a published proof has been found to be incorrect, the error almost always is found to lie concealed in such phrases. What seems obvious while writing the proof is not only *not* obvious, it may actually be false!

So where does that leave us with regard to PWWs? From the above comments it follows that in the strict formal sense of the word, a PWW is *not* a proof. Rather, it is a *suggestion of a proof*; it is an *outline of a proof*. Expressed another way, it is *proof cast in a poetic metaphor*. In a PWW, there is a kind of non-verbal communication going on between the author and the reader, and in that communication lie enough hints for the entire proof to be reconstructed. This means that a PWW depends on a shared culture of mathematics for its meaning: there is a common language being used by the author and the reader, a common lexicon or vocabulary. Without such a shared base, the PWW would be incomprehensible.

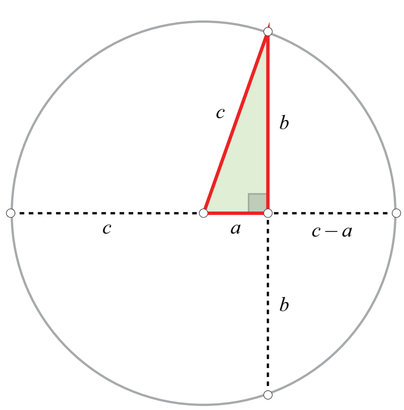
Viewed against the backdrop of such comments, there seems no substantive reason for not regarding a PWW as a proof. Accordingly, we shall accept the descriptions given above by [5] and [6].

We give below a few PWWs which are of particular elegance, along with their sources (when available). We hope that they will convince any sceptical reader of the value and worth of the PWW as a valid genre of proof.

A gallery of proofs without words

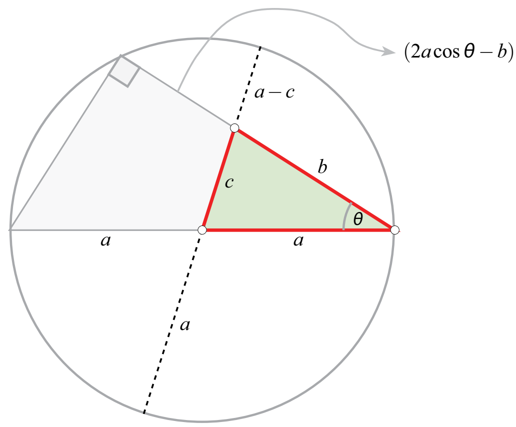
The theorem of Pythagoras. We start (naturally enough) with the venerable theorem of Pythagoras. We have featured the famous twelfth century "Behold!" proof of this theorem (due to Bhāskara II) in an earlier issue of *At Right Angles*, so we do not repeat it here. Instead we present a proof based on circle properties — specifically, the intersecting chords theorem, also called the 'crossed chords theorem'. It has been adapted from [2], page 8. See Figure 1.

The cosine rule. A small adaptation of the PWW for the theorem of Pythagoras yields a PWW for the cosine rule; it too draws from the intersecting



$$(c+a)\times(c-a) = b\times b$$
$$\therefore\ c^2-a^2=b^2$$
$$\therefore\ a^2+b^2=c^2$$

Figure 1. PWW for the theorem of Pythagoras



$$(a+c)\times(a-c)=b\times(2a\cos\theta-b)$$
$$\therefore\ a^2-c^2=2ab\cos\theta-b^2$$
$$\therefore\ c^2=a^2+b^2-2ab\cos\theta$$

Figure 2. PWW for the cosine rule

chords theorem for its inspiration. This PWW is from [2], page 32. See Figure 2.

The tangent of 15 degrees. What is the value of tan 15°? Figure 3 (if we have drawn it properly, and if this PWW is as effective as it claims to be) should reveal the answer! Namely, it should convince you that $\tan 15^\circ = 2-\sqrt{3}$. Please study the figure carefully, and let us know if it has persuaded you to agree with the statement.

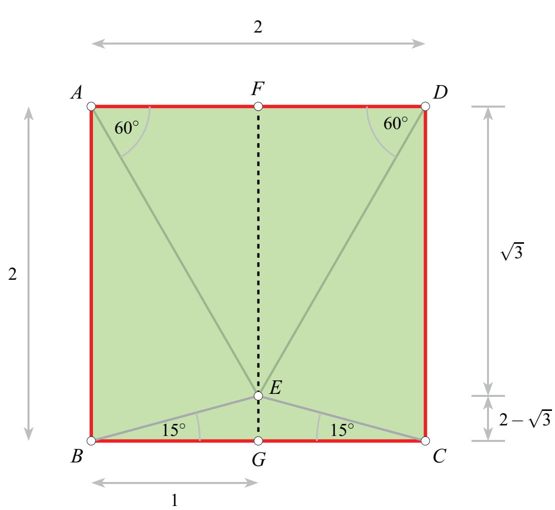


Figure 3. PWW to show that $\tan 15^\circ = 2-\sqrt{3}$

Triangular number identity. The triangular numbers T_n ('T-numbers') are defined to be the partial sums of the sequence of natural numbers $1, 2, 3, 4, \ldots$ (so they are the numbers 1, $1+2=3$, $1+2+3=6$, $1+2+3+4=10$, ...). They are generated by the formula

$$T_n = \frac{n(n+1)}{2}.$$

The T-numbers exhibit a large number of identities which are closely intertwined with properties of the square numbers. Among the simplest and most charming of these are: (a) The sum of two consecutive T-numbers is a perfect square. (b) If you multiply a T-number by 8 and add 1 to the result, you get a perfect square. There are nice PWWs for both these properties which we leave you to find. For now we present a PWW for a less obvious and much less well-known result, taken from [2], page 104. Here is the result itself:

$$3T_n + T_{n-1} = T_{2n}.$$

The PWW is depicted in Figure 4. Note that the figure has been drawn for the specific case $n=5$, so it only shows that $3T_5 + T_4 = T_{10}$. But it generalizes in a fairly obvious way to show that $3T_n + T_{n-1} = T_{2n}$ for all positive integers n.

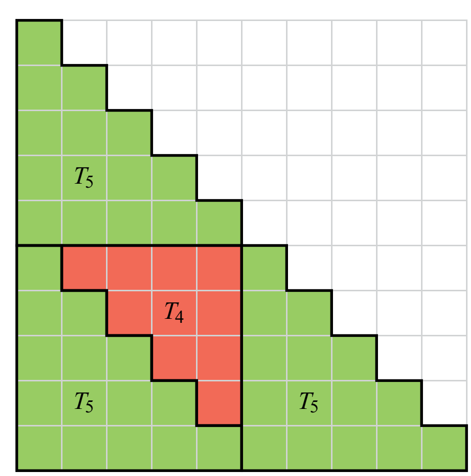
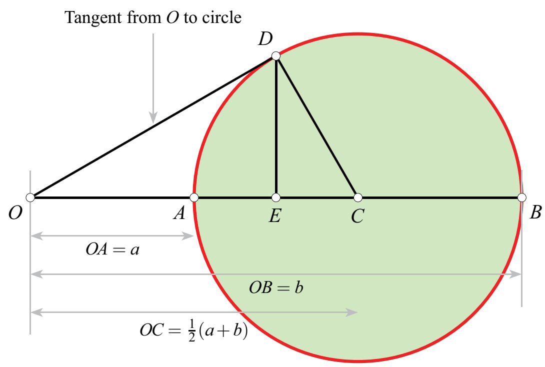


Figure 4. PWW to show that $3T_5 + T_4 = T_{10}$

In this PWW we see a theme which is very common in PWWs for number relations: *the PWW is shown only for a specific number*. But the way it is drawn gives a clear suggestion how it can be drawn for any number. *The passage to generalization is implicit in the way the figure is drawn.*



- $OD^2 = OA \times OB$
- $OD = \sqrt{ab}$
- $\frac{OE}{OD} = \frac{OD}{OC} \quad (= \cos\measuredangle DOE)$
- $OE = \frac{OD^2}{OC} = \frac{2ab}{a+b}$
- $OE =$ harmonic mean of a, b
- $OD =$ geometric mean of a, b
- $OC =$ arithmetic mean of a, b
- $OE < OD < OC$

Figure 5. PWW for the AM-GM-HM inequality

The AM-GM-HM inequality. We close this anthology with a PWW for the AM-GM-HM inequality, which plays a significant role in the article by Hussen elsewhere in this issue, *Simple Formulas for Square Roots*. The "AM-GM-HM inequality" is the statement that for any two positive numbers a and b, we have AM ≥ GM ≥ HM, where AM, GM and HM denote the arithmetic mean, the geometric mean and the harmonic mean respectively of a and b:

$$\text{AM} = \frac{a+b}{2},$$

$$\text{GM} = \sqrt{ab},$$

$$\text{HM} = \frac{2ab}{a+b}.$$

Moreover, equality holds precisely when $a = b$. The configuration depicted in Figure 5 demonstrates the property in a beautiful and succinct manner. (It has been drawn assuming that $a < b$.) Strictly speaking, this is *not* a PWW, as the derivations have been shown at the right side! But we have included it here as the figure puts the inequality into the framework of geometry in such a nice way.

References

[1] Bogomolny, A. Proofs Without Words. From "Interactive Mathematics Miscellany and Puzzles", http://www.cut-the-knot.org/ctk/pww.shtml, Accessed 18 November 2014

[2] Nelsen, Roger B. *Proofs Without Words: Exercises in Visual Thinking*. MAA

[3] Nelsen, Roger B. *Proofs Without Words II: More Exercises in Visual Thinking*. MAA

[4] Proofs without words. http://mathoverflow.net/questions/8846/proofs-without-words

[5] https://en.wikipedia.org/wiki/Proof_without_words

[6] http://mathworld.wolfram.com/ProofwithoutWords.html

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Simple formulas …

Approximating Square Roots and Cube Roots

Do they work?

feature

The idea of equivalent geometric forms is used in this study to devise simple formulas to estimate the square root and the cube root of an arbitrary positive number. The resulting formulas are easy to use and they don't take much time to calculate. They give excellent estimates which compare very favourably with those given by a pocket calculator.

Ali Ibrahim Hussen

Introductory Remarks

Historically the Babylonians were the first to devise an iterative method for computing square roots of numbers using rational operations that are easy to carry out ([1], [2]). The ancient Chinese method (200 BCE), commented on by Liu Hui in the third century CE, is similar in procedure to the long division method used in schools even today ([2]). The Greek mathematician Heron of Alexandria, who gave the first explicit description of the Babylonians iterative method, also devised a method for cube root calculation in the first century CE ([3]). In 499 CE, Aryabhata (Indian mathematician and astronomer) gave a method for computing cube roots of numbers of arbitrary size ([5]). In the sixteenth century, Isaac Newton devised an iterative method used to calculate square and cube roots of arbitrary numbers.

The method used in this article is different from the methods mentioned above. It is based on the idea of changing a geometric

Keywords: *Square root, cube root, estimation, accuracy, geometric mean, harmonic mean, inequality, square, cube*

figure or form into an equivalent 'regular' one with equal area or volume, e.g. changing a rectangle into a square, or a rectangular cuboid into a cube.

The idea was used by ancient Indian engineers who worked with designing and constructing temples (600 BCE). They used a practical method for changing a rectangle into a square. The method appears in the *Sulba-sutras* by Baudhayana (Sulba-sutras means the 'Rule of the Chord') ([6]). The idea is used also by David W. Henderson in his attempt to solve quadratic and cubic equations geometrically ([7]). In our case an algebraic analysis is performed instead of a geometric one.

1. Estimation of Square Roots

Let n be the given number whose square root is required. (Here, of course, n is positive.) We start by expressing n as a product ab, with a as close to b as possible (the closer the better). The task of computing $\sqrt{n} = \sqrt{ab}$ may be expressed geometrically as: Construct a square whose area is equal to that of a rectangle with dimensions $a \times b$ (Figure 1).

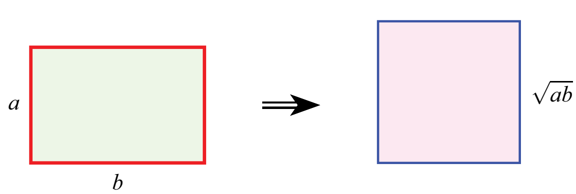


Figure 1.

Now note that $\sqrt{ab}$ is the *geometric mean* (GM) of the quantities a and b. We may try to approximate the GM by the *arithmetic mean* (AM) of a and b. It is well known that the AM exceeds the GM (strictly) if the numbers are unequal. But if the numbers are close to one another, then the AM is quite close to the GM.

Geometrically this may be viewed as follows. Assume that $a < b$. Within the $a \times b$ rectangle we draw a square on the shorter side a, then slice the leftover portion into two equal halves and stack them (one each) on to two adjacent sides of the square (Figure 2). The resulting shape is a square of side $\frac{1}{2}(a+b)$ minus a small piece in the corner, which is a square of side $\frac{1}{2}|a-b|$. If a is close to b, then this portion will be very small. So if we ignore the little square, the larger square may be taken to be the one we seek.

First approach to approximating square roots. The above reasoning gives us our *first algorithm* for approximating the square root of a given positive number n.

Algorithm I

Step 1: Write n as $a \times b$ where a and b are close to each other.

Step 2: Then our estimate of $\sqrt{n}$ is $\frac{1}{2}(a+b)$.

The closer a and b are to each other, the better will be our estimate.

This algorithm gives us fairly good results provided we are wise in our initial choice of numbers. The following examples illustrate this.

Example 1. Let $n = 6$. Let us take $a = 2.4$ and $b = 2.5$. Then our estimate for $\sqrt{6}$ is $\frac{1}{2}(2.4 + 2.5) = 2.45$. This may be compared with the actual value which is roughly 2.4495 (an error of about 0.02%).

Example 2. Let $n = 10$. Let us take $a = 3$ and $b = 10/3$. Then our estimate for $\sqrt{10}$ is $\frac{1}{2}(3 + 10/3) = 19/6 \approx 3.167$. This may be compared with the actual value which is roughly 3.162 (an error of about 0.14%).

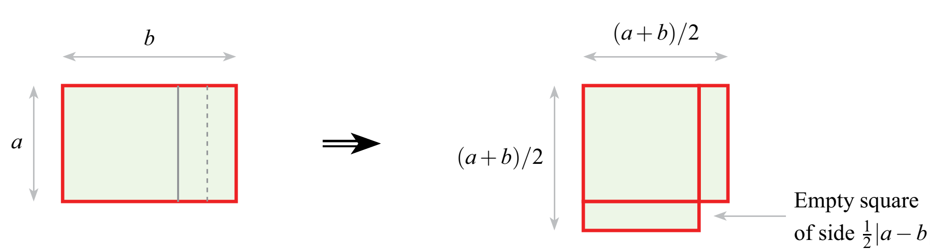


Figure 2.

We can do better that this by taking $a = 3.1$ and $b = 10/3.1$. Then our estimate for $\sqrt{10}$ is:

$$\frac{1}{2}\left(3.1 + \frac{10}{3.1}\right) = \frac{1}{2}\left(\frac{9.61 + 10}{3.1}\right) = \frac{19.61}{6.2} \approx 3.163.$$

The error now is about 0.02%.

Second approach to approximating square roots. In the study of statistics we encounter several different kinds of means defined for any given pair of positive numbers a and b. We have the **arithmetic mean** (AM), the **geometric mean** (GM) and the **harmonic mean** (HM), among many others (not listed here). These are defined as follows:

$$\text{AM} = \frac{1}{2}(a+b), \qquad \text{GM} = \sqrt{ab}, \qquad \text{HM} = \frac{2ab}{a+b}.$$

Each of these has its significance and uses. Of particular interest to us is the fact that if $a \neq b$, then:

$$\text{HM} < \text{GM} < \text{AM}.$$

These inequalities are never violated. The HM and AM *always* lie on opposite sides of the GM.

For example, if $a = 2$ and $b = 8$, then

$$\text{AM} = \frac{1}{2}(2 + 8) = 5, \qquad \text{GM} = \sqrt{2 \times 8} = 4,$$
$$\text{HM} = \frac{2 \times 2 \times 8}{2 + 8} = 3.2,$$

and, of course, $3.2 < 4 < 5$.

Recall that in our approach we start by selecting positive numbers a and b such that $n = ab$. We wish to compute the GM of a and b. But as the GM is difficult to compute, we choose to compute the AM instead. We could also choose to compute the HM (this too is an easy computation). Whichever one we choose to compute will then be our estimate for the GM.

If we use the AM, we always get an *over-estimate* for the GM. And if we use the HM, we always get an *under-estimate*.

Now an interesting idea strikes us: why not compute *both* the AM and HM, and then take their average? Their respective errors may then just cancel each other, and we may just get an estimate quite close to the GM. To our surprise, we find that this idea works very well.

So here is our *second algorithm* for approximating the square root of a given positive number n.

Algorithm II

Step 1: Write n as $a \times b$ where a and b are reasonably close to each other.

Step 2: Compute the AM and the HM of a and b.

Step 3: Compute the average of the AM and HM computed in Step 2. That is, compute:

$$\frac{1}{2}\left(\frac{a+b}{2} + \frac{2ab}{a+b}\right).$$

This is our estimate for $\sqrt{n}$.

This algorithm gives us extremely good results — far better than we would expect! The following examples illustrate this.

Example 3. Let $n = 6$. Let us take $a = 2.4$ and $b = 2.5$. Then our estimate for $\sqrt{6}$ is:

$$\frac{1}{2}\left(\frac{2.4 + 2.5}{2} + \frac{2 \times 2.4 \times 2.5}{2.4 + 2.5}\right) = \frac{1}{2}\left(2.45 + \frac{12}{4.9}\right) = \frac{4801}{1960} \approx 2.449489795.$$

Compare this with the true value: $\sqrt{6} = 2.449489742\ldots$. The estimate is correct to seven decimal places.

Example 4. Let $n = 10$. Let us take $a = 3$ and $b = 10/3$. Then our estimate for $\sqrt{10}$ is:

$$\frac{1}{2}\left(\frac{3 + 10/3}{2} + \frac{2 \times 3 \times 10/3}{3 + 10/3}\right) = \frac{1}{2}\left(\frac{19}{6} + \frac{60}{19}\right) = \frac{721}{228} \approx 3.16228.$$

This estimate is correct to four decimal places. (The true value is 3.162277660 ....)

Example 5. Let $n = 20$. Let us take $a = 4.5 = 9/2$ and $b = 20/4.5 = 40/9$. Then our estimate for $\sqrt{20}$ is:

$$\frac{1}{2}\left(\frac{9/2 + 40/9}{2} + \frac{2 \times 9/2 \times 40/9}{9/2 + 40/9}\right) = \frac{1}{2}\left(\frac{161}{36} + \frac{720}{161}\right) = \frac{51841}{11592} \approx 4.472135955.$$

This estimate is correct to eight decimal places.

2. Estimation of Cube Roots

Let n be the given number whose cube root is required. We start by expressing n as a product abc, with a, b and c as close as possible to each other (the closer the better; here, of course, $a, b, c > 0$). The task of computing $n^{1/3} = (abc)^{1/3}$ may be expressed as: Compute the geometric mean (GM) of a, b, c. This task may be given a geometric form as follows: Construct a cube whose volume is equal to that of a cuboid with dimensions $a \times b \times c$.

As we did with the square root, we start with the arithmetic mean (AM) of a, b, c and then see how we can 'improve' it. Let $d = \frac{1}{3}(a + b + c)$ be the AM. It is well known that the AM exceeds the GM if the numbers are unequal. So let us consider subtracting a suitably small quantity h from d. We must find h so that $(d - h)^3 = n$. Write $x = d - h$, so x is going to be our estimate for the desired cube root. We argue as follows:

$$n = d^3 - 3d^2h + 3dh^2 - h^3,$$

$$\therefore\ d^3 - n \approx 3d^2h - 3dh^2$$

$$\text{(we drop the } h^3 \text{ term since } h \approx 0),$$

$$\therefore\ d^3 - n \approx 3dh(d - h),$$

$$\therefore\ \frac{d^3 - n}{3d} \approx x(d - x)$$

$$\text{(since } d - h = x \text{ and } h = d - x).$$

In the last line we change the approximation sign to an equality sign and solve the resulting equation — which is now a *quadratic equation*, not a cubic equation, and therefore easy to solve — for x. The answer we get will be our estimate for the cube root of n.

Example 6. Let us estimate the cube root of $n = 6$. Write $6 = 1 \times 2 \times 3$. Our initial estimate for the cube root of 6 is the AM of 1, 2, 3, i.e., $d = \frac{1}{3}(1 + 2 + 3) = 2$. The value of $\left(d^3 - n\right)/(3d)$ is:

$$\frac{2^3 - 6}{3 \times 2} = \frac{1}{3}$$

Hence the equation we must solve is:

$$x\,(2 - x) = \frac{1}{3}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = 1 \pm \frac{\sqrt{6}}{3}.$$

We must use the positive root. So our estimate for the cube root of 6 is:

$$1 + \frac{\sqrt{6}}{3} \approx 1.8165.$$

Here is the actual cube root, computed using a calculator:

$$6^{1/3} \approx 1.8171.$$

Our estimate is correct to two decimal places.

We can do better by writing $6 = 2 \times 2 \times 3/2$. Then we have:

$$d = \frac{1}{3}\left(2 + 2 + \frac{3}{2}\right) = \frac{11}{6},$$

$$\frac{d^3 - n}{3d} = \frac{35}{1188}.$$

Hence the equation we must solve is:

$$x\left(\frac{11}{6} - x\right) = \frac{35}{1188}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = \frac{11}{12} \pm \frac{\sqrt{127149}}{396}.$$

We must use the positive root. So our estimate for the cube root of 6 is:

$$x = \frac{11}{12} + \frac{\sqrt{127149}}{396} \approx 1.817120162.$$

This is accurate to six decimal places.

Example 7. Let us estimate the cube root of 10. Write $10 = 2 \times 2 \times 5/2$. Our initial estimate for the cube root of 10 is the AM of $2, 2, 5/2$, i.e., $d = \frac{1}{3}(2 + 2 + 5/2) = 13/6$. The value of $\left(d^3 - n\right)/(3d)$ is:

$$\frac{(13/6)^3 - 10}{13/2} = \frac{37}{1404}$$

Hence the equation we must solve is:

$$x\left(\frac{13}{6} - x\right) = \frac{37}{1404}.$$

Using the quadratic formula we find that the roots of this equation are:

$$x = \frac{13}{12} \pm \frac{\sqrt{251277}}{468}.$$

We must use the positive root. So our estimate for the cube root of 10 is:

$$\frac{13}{12} + \frac{\sqrt{251277}}{468} \approx 2.154434558.$$

Here is the actual cube root, computed using a calculator:

$$10^{1/3} \approx 2.154434690.$$

Our estimate is correct to six decimal places.

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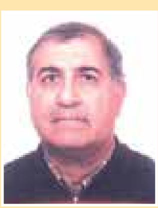

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We retrieved this snippet from http://www.futilitycloset.com/2014/11/04/all-right-then-2/:

All Right Then

W.H. Auden won first prize for mathematics at St. Edmund's School in Hindhead, Surrey, when he was 13. He recalled being asked to learn the following mnemonic around 1919:

Minus times Minus equals Plus;
The reason for this we need not discuss.

At Right Angles is interested in knowing how you teach this concept.
Teachers cannot get away from discussion now. So how do you explain this rule?
Do write to us at AtRiA.editor@apu.edu.in.

The best responses will be published in the next issue.

feature

How do we explain their success?

Understanding the Formulas

High-school math to the rescue …

In the accompanying article Approximating Square Roots and Cube Roots, *the author Ali Ibrahim Hussen has proposed easy-to-use formulas for finding approximate values of the square root and cube root of an arbitrary positive number n. The formulas are found to give fairly satisfactory results, as measured by the low percentage error. In this article we explain mathematically why this is so.*

SHAILESH SHIRALI

Square root

Both methods start by writing n as $a \times b$, where a and b are close to each other. Now we have the following:

$$\sqrt{n} = \sqrt{ab} = \sqrt{a \times a \times \frac{b}{a}} = a\sqrt{\frac{b}{a}}.$$

Let $b/a = 1 + x$. The fact that "b is close to a" translates to: "$1 + x$ is close to 1", i.e., "x is close to 0". This is generally written as: $x \approx 0$. (Another way of writing it is: $x \ll 1$.) With this notation we have: $\sqrt{n} = a\sqrt{1+x}$.

Accordingly, it is sufficient if we study what the two algorithms yield for the value of $\sqrt{1+x}$.

Keywords: *Square root, cube root, estimation, accuracy, geometric mean, harmonic mean, inequality, square, cube*

First approximation to the square root. We write $1 + x = 1 \times (1 + x)$. What we need is the geometric mean (GM) of 1 and $1 + x$, but we approximate the GM by the arithmetic mean (AM), i.e., by $(1 + 1 + x)/2 = 1 + x/2$. Hence the first approximation to the desired square root is:

$$\sqrt{1+x} \approx 1 + \frac{x}{2}. \qquad (1)$$

In analytic terms this approximation is well known and easy to understand, for it is the *linear approximation* to the function $\sqrt{1+x}$ as given by the binomial theorem. Recall the statement of the binomial theorem for exponents n other than positive integers:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \cdots,$$

where $|x| < 1$ for convergence of the infinite series on the right side. (We must have such a condition, because if n is not a positive integer then the series does not terminate. If however n is a positive integer, the expression on the right side is simply a polynomial in x, of degree n, so the question of convergence does not arise, and the statement is then an identity, valid for all x.)

For the particular case $n = 1/2$, we get the series for the square root of $1 + x$:

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots. \qquad (2)$$

If x is close to 0, then we may drop all terms with degree 2 or more, on the ground that they are small and do not make much of a difference to the total. The resulting formula, $1 + x/2$, is called the "linear approximation" to $\sqrt{1+x}$. Thus the first approximation presented in Hussen's article is equivalent to the linear approximation. Table 1 shows how well this formula does, and Figure 1 displays the same relationship in graphical terms.

We see that if x is close to 0, the linear approximation gives good results. Obviously, better than the linear approximation is the quadratic approximation, $1 + x/2 - x^2/8$, and still better is the cubic approximation. But we leave the testing of these formulas to the reader.

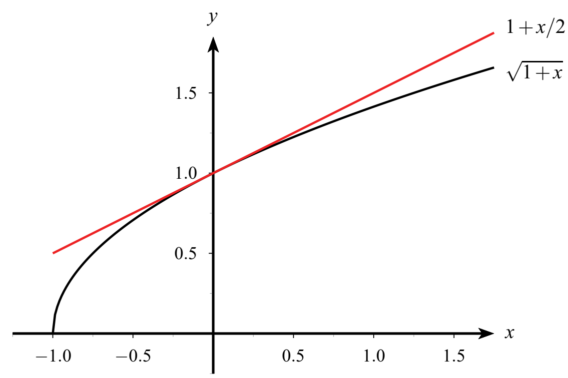

Figure 1. Graphical study of the approximation $\sqrt{1+x} \approx 1 + x/2$

Second approximation to the square root. As earlier we write $1 + x = 1 \times (1 + x)$. Now we approximate the GM of 1 and $1 + x$ (which is what is required) by the average of the AM and the harmonic mean (HM) of 1 and $1 + x$. We have:

$$\text{Arithmetic mean (AM)} = \frac{2+x}{2},$$

$$\text{Harmonic mean (HM)} = \frac{2(1+x)}{1+(1+x)} = \frac{2(1+x)}{2+x},$$

$$\text{Average of AM and HM} = \frac{1}{2}\left(\frac{2+x}{2} + \frac{2+2x}{2+x}\right).$$

The last expression when simplified yields a fresh estimate for the square root of $1 + x$:

$$\frac{1 + x + x^2/8}{1 + x/2}. \qquad (3)$$

x	1	0.2	0.1	0.01	0.001
$1 + x/2$	1.5	1.1	1.05	1.005	1.0005
$\sqrt{1+x}$	1.414	1.0954	1.0488	1.004987	1.00049987
% error	6.065	0.4158	0.1135	0.00124	0.0000126

Table 1. Error study of the first approximation

x	1	0.2	0.1	0.01	0.001
$(\mathrm{AM}+\mathrm{HM})/2$	1.41667	1.09545	1.04881	1.00499	1.0005
$\sqrt{1+x}$	1.414	1.0954	1.0488	1.004987	1.00049987
% error	0.173	0.00086	0.000064	7.66×10^{-9}	7.77×10^{-13}

Table 2. Error study of the second approximation

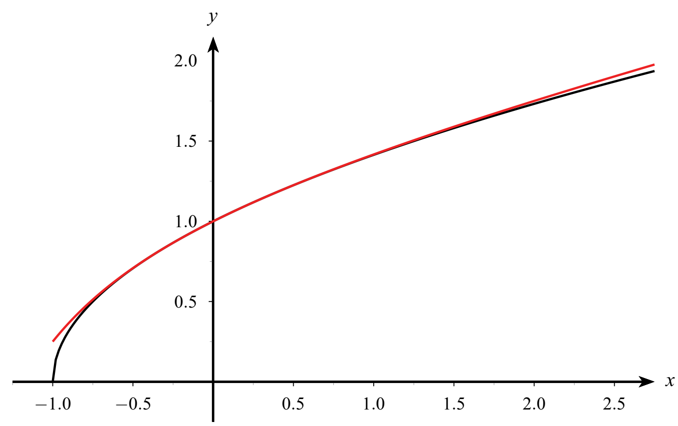

Figure 2. Graphical study of the approximation $\sqrt{1+x} \approx 1/2\,[1 + x/2 + 2(1+x)/(2+x)]$

Table 2 shows how well the new formula does, while Figure 2 displays the same relationship in graphical terms. Note the closeness of the two graphs even for values of x that one would not consider 'small'. We see that the second approximation does significantly better than the first formula — far better than one would ever have expected.

Insight into why this formula does so well comes when we examine the relevant power series expansion, which we get using the binomial theorem. We have:

$$\frac{1+x+x^2/8}{1+x/2} = \left(1+x+\frac{x^2}{8}\right)\left(1+\frac{x}{2}\right)^{-1}$$

$$= \left(1+x+\frac{x^2}{8}\right)\left(1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{16}-\cdots\right)$$

$$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^4}{32}+\cdots$$

(on multiplying out, term by term). (4)

Compare this with:

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots.$$

We see that the two series coincide right up to the x^3 term (and even their x^4 coefficients are quite close to each other). No wonder that the two formulas agree so closely.

Cube root

Following the discussion in the previous section, it is sufficient if we focus on estimating the cube root of $1+x$ where $x \approx 0$. The proposed method starts by expressing $1+x$ as a product of three numbers close to each other. We shall write $n = 1 \times 1 \times (1+x)$, i.e., our three numbers are $1, 1, 1+x$. Then the task of computing $(1+x)^{1/3}$ is equivalent to: Compute the geometric mean (GM) of $1, 1, 1+x$.

As we did with the square root, we start with the arithmetic mean of these three numbers; we get $d = 1 + x/3$. Now we try to improve this estimate by subtracting some quantity h from d such that $(d-h)^3 = 1+x$. Write $e = d - h = 1 + x/3 - h$, so e is going to be our new estimate for the desired cube root. Note that $h = 1 + x/3 - e$. We argue as follows:

$$1 + x = d^3 - 3d^2h + 3dh^2 - h^3,$$

$$\therefore\ d^3 - 1 - x \approx 3d^2h - 3dh^2$$

(we drop the h^3 term since $h \approx 0$),

$$\therefore\ d^3 - 1 - x \approx 3dh(d-h),$$
$$\therefore\ \frac{d^3-1-x}{3d} \approx h(d-h),$$
$$\therefore\ \frac{(1+x/3)^3-1-x}{3+x} \approx e\left(1+\frac{x}{3}-e\right). \qquad (5)$$

The equation in the last line must be solved for e to give us our estimate for $(1+x)^{1/3}$. As the equation is quadratic, the manipulations can be done using known formulas. The algebra is tedious, but after some work (and a fair bit of help from a reliable computer algebra system like *Mathematica*!) we get the following result:

$$e = \frac{\left(27+18x+3x^2 + \sqrt{3\,(243+324x+54x^2-12x^3-x^4)}\right)}{18(3+x)}. \qquad (6)$$

Now we must expand this expression into an infinite series, and then we need to compare the result with the binomial expansion of $(1+x)^{1/3}$. The binomial expansion for the cube root is easily found, using the binomial theorem:

$$(1+x)^{1/3} = 1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}-\frac{10x^4}{243} + \frac{22x^5}{729}-\frac{154x^6}{6561}+\frac{374x^7}{19683}+\cdots. \qquad (7)$$

Generating the corresponding series for the expression e involves a lot more work, and this time we very definitely need the services of *Mathematica* to simplify the expressions involved. Here is what we get:

$$e = 1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}-\frac{10x^4}{243}+\frac{22x^5}{729}-\frac{157x^6}{6561} + \frac{395x^7}{19683}+\cdots. \qquad (8)$$

The two series agree all the way till the fifth degree term, and even their sixth and seventh degree terms do not differ by very much. Well! Now we understand why the method gives such astonishingly good results.

Another algorithm for cube root

Following the success of the second algorithm for square root, we propose something similar for the cube root. It does not do nearly as well as Hussen's algorithm presented above. On the other hand, that algorithm requires us to compute a square root at one stage, and this diminishes its attractiveness somewhat. It would be nicer if we could get a cube root estimate without any square root computation! — i.e., by sticking only to rational operations. We present one such possibility here. It is about as simple-minded as any algorithm can be, yet it does quite well.

Simple-minded algorithm for cube root

Step 1: Write n as $a\times b\times c$ where a, b and c are close to each other.

Step 2: Compute the AM and the HM of a, b and c.

Step 3: Compute the average of the AM and HM computed in Step 2. This is our estimate for $n^{1/3}$.

For example, take $n = 10$. Write 10 as $10 = 2\times 2\times 5/2$. Then we have:

$$\text{AM} = \frac{2+2+5/2}{3} = \frac{13}{6},$$

and:

$$\text{HM} = \frac{3}{1/2+1/2+2/5} = \frac{3}{7/5} = \frac{15}{7}.$$

Hence our estimate for the cube root of 10 is:

$$\frac{1}{2}\left(\frac{13}{6}+\frac{15}{7}\right) = \frac{181}{84} \approx 2.1548.$$

Here is the actual value: $10^{1/3} = 2.1544$. That seems close enough given how little we worked for it!

To study it analytically, we check the series expansion it gives for the cube root of $1+x$, after writing it as $1\times 1\times(1+x)$. We have, for the numbers $1, 1, 1+x$:

$$\text{Arithmetic mean (AM)} = \frac{3+x}{3},$$

$$\text{Harmonic mean (HM)} = \frac{3}{1/1+1/1+1/(1+x)} = \frac{3(1+x)}{3+2x},$$

$$\text{Average of AM and HM} = \frac{9+9x+x^2}{9+6x}.$$

So our estimate for the cube root of $1 + x$ is

$$\frac{9 + 9x + x^2}{9 + 6x} = \frac{1 + x + x^2/9}{1 + 2x/3}. \qquad (9)$$

The series expansion for the last expression is:

$$\begin{aligned}\frac{1 + x + x^2/9}{1 + 2x/3} &= \left(1 + x + \frac{x^2}{9}\right) \\ &\quad \left(1 - \frac{2x}{3} + \frac{4x^2}{9} - \frac{8x^3}{27} + \cdots\right) \\ &= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{2x^3}{27} - \frac{4x^4}{81} + \cdots. \end{aligned} \qquad (10)$$

Comparing this with the binomial expansion of $(1 + x)^{1/3}$,

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \cdots, \qquad (11)$$

we see that the two series agree only till the x^2 term. The coefficients of the x^3 terms are close to each other but not the same, and likewise for the x^4 terms. This explains why the results of this recipe are only moderately good.

An algorithm for higher order roots

The idea proposed in the last section may be extended easily to k^{th} roots where k is an arbitrary positive integer.

Simple-minded algorithm for k^{th} root of $1+x$

Step 1: Write $1 + x$ as $\underbrace{1 \times 1 \times \cdots \times 1}_{(k-1)} \times (1 + x)$.

Step 2: Compute the AM and the HM of the list $\underbrace{1, 1, \ldots, 1}_{(k-1)}, 1 + x$:

$$\text{AM} = \frac{1 + 1 + \cdots + 1 + (1 + x)}{k} = \frac{k + x}{k},$$

$$\begin{aligned}\text{HM} &= \frac{k}{1/1 + 1/1 + \cdots + 1/1 + 1/(1 + x)} \\ &= \frac{k(1 + x)}{k + (k - 1)x}.\end{aligned}$$

Step 3: Compute the average of the AM and HM computed in Step 2. This is our estimate for $(1 + x)^{1/k}$:

$$(1 + x)^{1/k} \approx \frac{2k^2 + 2k^2x + (k - 1)x^2}{2k^2 + 2k(k - 1)x}. \qquad (12)$$

For example, take $x = 1/4$ and $k = 5$. That is, we want the fifth root of $5/4$, i.e., $(1.25)^{1/5}$. The formula yields:

$$\frac{50 + 50/4 + 1/4}{50 + 10} = \frac{251}{240} \approx 1.0458.$$

For comparison here is the actual value: $1.25^{1/5} = 1.0456$.

Closing remarks

Finding good rational approximations for irrational difficult-to-compute numbers is a theme that goes far back in history, and over the ages many remarkable algorithms have been devised. The venerable "long-division square root algorithm" is part of this tradition, as is a remarkable formula found by Bhāskara I to compute values of the sine function. (This will be the subject of a future article.) What is satisfying and of interest is that we are able to account for the good behaviour of the various algorithms in the accompanying article by Hussen. It reaffirms our faith in the subject!

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Of Art and Mathematics

Paradoxes:

True AND/OR False?

Part 1 of 2

feature

Punya Mishra & Gaurav Bhatnagar

This is the first sentence of this article.

Clearly the sentence above is true (not highly informative but true). Contrast this to the next sentence, below:

This is the first sentence of this article.

Now the second statement, though identical to the first, is clearly false.

Such sentences that speak about themselves are called *self-referential* sentences, because they are, in a way, looking at themselves in the mirror and describing themselves. Figure 1, is a design for the word "reference" so it looks the same when reflected in a mirror.

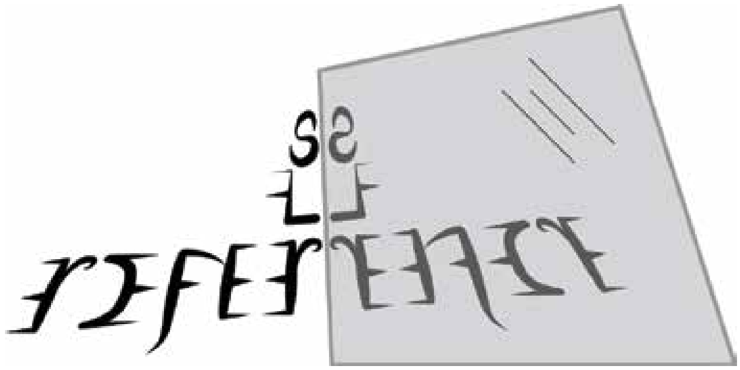

Figure 1. Self-reference looks in a mirror. The word "self-reference" is written in a manner that it looks the same when reflected in a mirror (a wall reflection).

Keywords: *Truth value, self-reference, paradox, axiom, theorem, consistency, circular argument, proof, Zeno, ourobouros, ambigram*

Figure 2. An ambigram for Paradox, the subject of this column

Such self-referential sentences sometimes lead to paradoxes, and paradoxes are the topic of this article. As usual we use the medium of ambigrams to communicate some of these paradoxical ideas (see Figure 2 for an ambigram of Paradox). And we produce some graphical paradoxes of our own for you to think about.

Mathematical Truth

To understand what self-referential statements have to do with mathematics we need to get a bit deeper into what mathematicians mean by the words *true* and *false*. A mathematical theory consists of a large number of statements. There are two special types of true statements in any mathematical theory–axioms and theorems.

For example, consider the development of plane geometry. We begin with certain axioms (such as: given a line and a point not on the line, there is exactly one line through that point parallel to the given line). Axioms are all considered to be true. Now by following the rules of logic, from Axioms one *proves* some other statements that are called theorems. If the proof is valid, we say the theorem is true. For example, a theorem is: The sum of three angles of a triangle is equal to two right angles. Each theorem is proved using the axioms, or the previously proved theorems. Figure 3 includes an ambigram of the word "axiom" that is then used over and over again to create an ambigram of the word "theorem."

Each statement in this theory is either true or false—it cannot be both, otherwise there will be a contradiction. And we will see shortly that contradictions are not allowed in mathematics.

Figure 3. Rotational ambigrams for the words "axiom" and "theorem"' – except that the word "theorem" is both an ambigram and constructed from the multiple axioms

Puzzle:

Can you decipher these strange squiggles below? Hint: There are two words related to this article

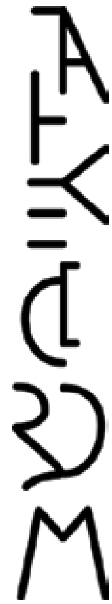

Figure 4. What do these squiggles mean?

In this theory, the axioms are taken to be true. However it is not necessary that the axioms are 'true' in every context. For example, the axioms of plane geometry are true in the idealized plane, but do not hold for the surface of the sphere, where 'lines' are simply *great circles*, which are formed by the intersection of the sphere with a plane passing through the center of the sphere. The equator, and lines of longitude are examples of great circles on a spherical globe. In this geometry, there is no line parallel to the given line from a point not on the line! This is because two great circles always meet. But surely the geometry of the sphere is equally "true" in the real world. (This kind of geometry, on the surface of the sphere, is called Riemannian Geometry).

What mathematical theories try to achieve is a consistency, where by consistency we mean: given the axioms and theorems proved within the theory (using the rules of logic), none of the statements contradict each other. Proofs are means to convince ourselves that the statements are "true" in the mathematical theory.

In developing a mathematical theory, one needs to be careful to avoid a circular proof. A circular proof is when the proof of a statement uses the statement itself! Figure 5 is a reflection chain ambigram of the word "proof" — a visual circular proof!

A circular argument can be difficult to find. Say in proving a statement P we use the truth of a

Figure 5. A visual representation of a circular proof! This design reads the same both at the front (as in red) or at the back — or even when read in a mirror.

statement Q. But the proof of the statement Q involves the statement P. A good example of circular reasoning is in the book Catch 22,

> "You mean there's a catch?"
>
> "Sure there's a catch", Doc Daneeka replied. "Catch-22. Anyone who wants to get out of combat duty isn't really crazy."
>
> There was only one catch and that was Catch-22, which specified that a concern for one's own safety in the face of dangers that were real and immediate was the process of a rational mind. Orr was crazy and could be grounded. All he had to do was ask; and as soon as he did, he would no longer be crazy and would have to fly more missions. Orr would be crazy to fly more missions and sane if he didn't, but if he was sane, he had to fly them. If he flew them, he was crazy and didn't have to; but if he didn't want to, he was sane and had to. Yossarian was moved very deeply by the absolute simplicity of this clause of Catch-22 and let out a respectful whistle.
>
> "That's some catch, that Catch-22," he observed.
>
> "It's the best there is," Doc Daneeka agreed.

Or in the character Tippler in the Little Prince who says he drinks so that he may forget that he is ashamed of drinking! As the little prince says,

> "The grown-ups are certainly very, very odd."

In mathematics circular proofs show up when something that is assumed is then used to prove the same thing. For instance here is a circular proof of the Pythagorean theorem.

> Let ΔABC be a right triangle with sides a, b, c. As usual, let c be the hypotenuse, the side opposite the right angle C. We know that $\sin B = b/c$ and $\cos B = a/c$.
>
> Now using the elementary trigonometric identity $\cos^2 B + \sin^2 B = 1$, we find that $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$, as required.

The only problem with this proof is that it presupposes the Pythagorean theorem—the very theorem that it sets out to establish.The proof of $\cos^2 B + \sin^2 B = 1$ relies on the Pythagorean Theorem! This is a good example of a vicious circle (see the design for *ouroboros*, Figure 6, for another, more lethal, variant of a vicious circle!).

Why do Mathematicians not allow any contradictions in the theories they build?

Mathematicians avoid contradictions because they can completely destroy the entire theory. This is because of a theorem of logic: *a false proposition implies any proposition*. Given that a false statement implies *any* statement, there is not much point in having a theory that has false

Figure 6. A chain rotation ambigram for the word "ouroboros" representing the idea of a snake eating its own tail. The idea of the ouroboros has recurred throughout history – such as the image in the middle, which is from a late medieval alchemical manuscript (courtesy Wikimedia Commons).

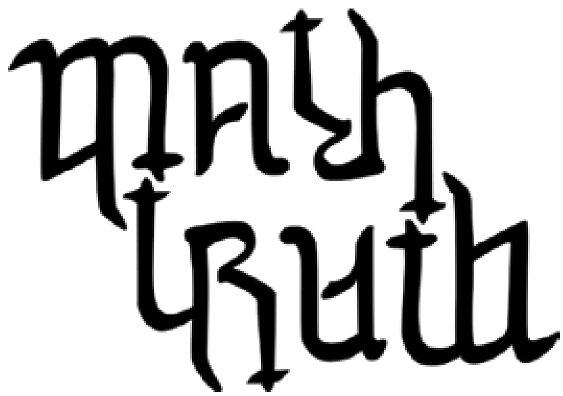

Figure 7. An ambigram about the relationship of math to truth

statements. For instance, on the one hand, we can prove a statement such as: There are an infinite number of prime numbers (as Euclid did over 2000 years ago). However, if even *one* false statement creeps into our mathematical universe, we can also prove that: There are only finitely many prime numbers! Or that there are exactly 317 prime numbers. Or that there are no prime numbers! Or that prime numbers are made of sweet buttermilk!

An example of a 'Proof' using a false proposition is this famous (probably apocryphal)story about the philosopher and mathematician Bertrand Russell (as retold by Raymond Smullyan in his classic book *What is the name of this book?).* Russell once told a dinner audience that "a false proposition implies any proposition." He was challenged to show that if $2 + 2 = 5$ (clearly a false statement) then he could prove that he (Russell) is the Pope. Russell then responded as follows:

> Given that $2 + 2 = 5$. Subtract 3 from both sides to get $1 = 2$. Now consider the following statements: The Pope and I are two. But $2 = 1$. So the Pope and I are one. Thus I am the Pope!

Note that starting from a false statement we end up with a nonsensical statement that "Russell is the Pope'. Thus something is wrong with the argument.

Mathematicians avoid contradictions like the plague (even more than writers avoid clichés). This is the reason why we insist on proofs in mathematics—to convince ourselves that all the statements are true. Figure 6 is a design where "math" rotates to read the word "truth."

Sometimes contradictions lead to paradoxes (or apparent paradoxes). Paradoxes are contradictory statements and have to be false. But since false statements are not allowed, there has to be some flaw in the reasoning. Resolving these paradoxes helps us understand the flaws in our reasoning. And more importantly, thinking about these paradoxical situations is fun.

Before we get into some serious self-contradictory paradoxes here is one that goes back a while – and one that turns out not to be a paradox if addressed with the right mathematical tools.

Zeno's Paradox

Zeno's paradoxes are about the impossibility of motion. A simple example is as follows. Suppose you have to go from a point A to a point B, which are 1 km distant from each other. Then first you have to reach halfway, a distance of half a km away. Then you have to go from mid point of AB (say A_1) to B. Again you have to first go half the distance A_1B which is one-fourth of a km. Next we have to go half the remaining distance, that is, one-eighth of a km. Going on in this fashion, Zeno asserted that we can never reach B. In other words, it is impossible to go from A to B. Thus Zeno showed by this argument that motion is impossible!

What is wrong with Zeno's argument? Zeno's paradoxes forced philosophers and mathematicians to think of the *continuum* and concepts such as infinite series. In our example above, we find that

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

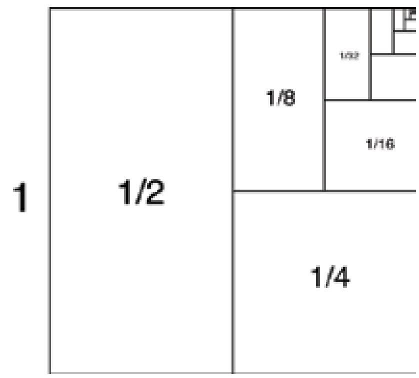

Figure 8. A ‘proof by pictures’ of the sum of the geometric series and how an infinite number of additions can lead to a finite sum

Figure 9. A visual Zeno Paradox, where “Zeno” gradually transforms to “Zero” – where the letter “n” changes step by step to the letter “r.” Is Zeno ever Zero?

which follows from the formula for the sum of the geometric series. Figure 8 shows a “proof by pictures” of this series. We can use the concept of infinite series to resolve Zeno’s paradox, by noting that the sum of an infinite number of additions can be a finite quantity.

Figure 9 shows an ambigrammatic approach to Zeno’s paradox; here the word Zeno tends to Zero!

In the Geometric Series, the infinite sum is a finite quantity. The ambigram of Figure 10 is about the word “Finite” written in such a manner that it becomes the symbol for infinity!

Figure 10. Finite reflection in a circle. The word finite repeats in a circle – and is also reflected in a mirror. Taken together the main image and its reflection from the symbol for infinity.

In conclusion

With this we come to the end of our first part of our reconnaissance of the domain of paradoxes in mathematics. There is a lot more to come…but for that you will have to wait for part 2 of this article.

So with that, we should let you know that though it may seem that way, *this* sentence is surely not the last word on the topic. This is. No. This. Word.

Answer to puzzle: If you place a mirror vertically along the middle of the squiggles you will see two words – Axiom and Theorem (as follows).

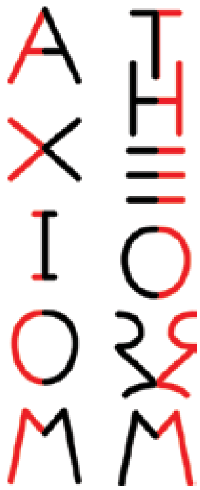

Figure 11. Solution to Puzzle 1

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Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav (except when mentioned otherwise). Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.

in the classroom

Generalization and Specialization

The Strange Case of the Pythagorean Theorem

Shailesh Shirali

The word 'generalize' is extremely dear to mathematicians; they are always looking for ways to generalize something or the other! This should not come as a surprise, because generalization is utterly basic to mathematics. (It is just as basic to the notion of language, but we won't go into that here.) In this article we examine what 'generalization' means, along with its complementary action, 'specialization'. Using a few simple examples, we show that even in very elementary contexts there lurk strange paradoxes.

What does it mean to 'generalize'?

Say we have a proposition Q which contains some free variables or parameters. When those parameters are given particular values, or some constraints are placed on them, let the proposition take on a new form P. *Then,* P *is said to be a* ***specialization*** *of* Q, *and* Q *is said to be a* ***generalization*** *of* P.

(1) Consider the following pair of propositions:

(P) $10^2 - 1 = 9 \times 11$.

(Q) $n^2 - 1 = (n-1)(n+1)$ for all numbers n.

P is clearly a particular case of Q in which n has been given the value 10; so P is a specialization of Q, while Q is a generalization of P.

(2) Consider the following pair of propositions:

(P) The area of a circle with radius r is πr^2.

(Q) The area of an ellipse with semi-axes a and b is πab.

P is a particular case of Q in which the semi-axes have equal length r (this is so because the circle is a special case of an ellipse; see Figure 1); hence P is a specialization of Q, while Q is a generalization of P.

(3) Consider the following statements P and Q which refer to an arbitrary positive integer n and the remainder when n is divided by 9. The symbol $s(n)$ denotes the sum of the digits of n when it is written in base ten.

(P) n is divisible by 9 if and only if $s(n)$ is divisible by 9.

(Q) The remainder when n is divided by 9 is the same as the remainder when $s(n)$ is divided by 9.

Statement P is the familiar test for divisibility by 9. You may not be familiar with statement Q, but it is true and is proved in exactly the same way that P is proved. Here is an illustration of Q in action: if $n = 175$, then $s(n) = 13$. Please check that 175 and 13 leave the same remainder (namely, 4) under division by 9.

Here, P is a special case of Q, for it corresponds to the case when the remainder is 0. So P is a specialization of Q, while Q is a generalization of P. (Loosely speaking, "Q contains 9 times as much information as P".)

(4) Consider the following pair of statements which refer to a circle ω with centre O and distinct points A, B, C on ω.

(P) If BC is a diameter of ω, then $\angle BAC$ is a right angle; see Figure 2 (a).

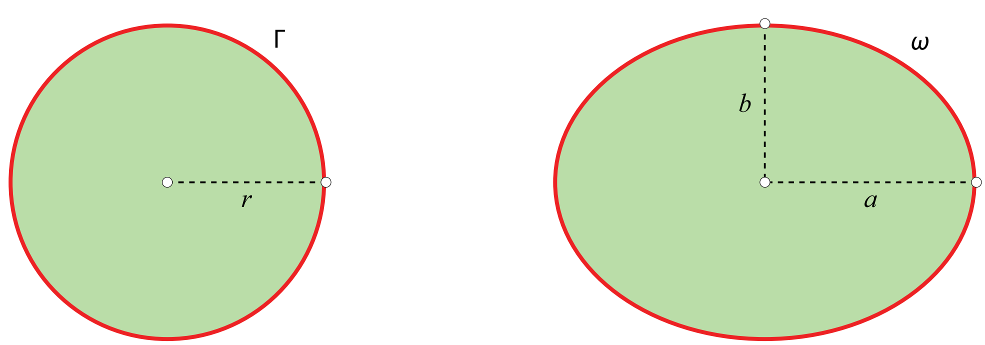
(Q) If A and O lie on the same side of BC, then $\angle BOC = 2\angle BAC$; see Figure 2 (b).

Here, P is a special case of Q in which $\angle BOC = 180°$.

(5) Consider the following statements which refer to a convex quadrilateral $ABCD$ with sides $AB = a, BC = b, CD = c, DA = d$, semi-perimeter $s = (a + b + c + d)/2$ and area Δ.

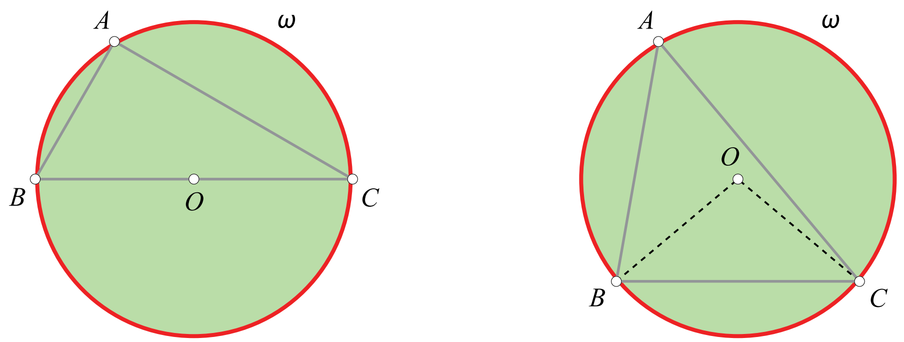
(P) If quadrilateral $ABCD$ is cyclic, its area Δ is given by

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d).$$



(a): Circle with radius r (b): Ellipse with semi-axes a, b

Figure 1. A circle may be regarded as a particular case of an ellipse



(a): The case when $\measuredangle A = 90^\circ$ (b): The general case

Figure 2. Two circle theorems: (a) is a particular case of (b)

(Q) If $ABCD$ is a general quadrilateral, its area Δ is given by

$$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2 \tfrac{1}{2}(A+C).$$

Here, P is a special case of Q; for, if the quadrilateral is cyclic, then $A + C = 180^\circ$, so the cosine term in Q vanishes (because $\cos 90^\circ = 0$) and we get formula P.

Some of you may know that P was first found by Brahmagupta.

Note that Q implies much more than P. Here is a lovely corollary of Q which derives from the fact that the cosine term comes in squared form and so is never negative: If the sides a, b, c, d of a quadrilateral are fixed, then its area is largest when the quadrilateral is cyclic.

Three Apparent Generalizations Of The Pythagorean Theorem

The title of this section is "Three Apparent Generalizations Of The PT" (we use the short-form 'PT' for 'Pythagorean Theorem'). We should add the following as a subtitle: "Or Are They 'Mere' Corollaries?" We present three propositions which can be thought of as generalizations of the PT, but which have an odd, paradoxical feature associated with them. The first one is the **cosine rule**.

Theorem 1 (Cosine rule). In $\triangle ABC$, we have the following relationship:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

The cosine rule clearly contains the PT as a special case: if $\angle A = 90^\circ$ then $\cos A = 0$, therefore $a^2 = b^2 + c^2$.

It also implies the *converse* of the PT; for, the cosine of an angle of a triangle is 0 precisely when the angle is a right angle. Hence if it happens that $a^2 = b^2 + c^2$, then it must be that $\cos A = 0$ (since bc is not zero), and therefore that $\angle A = 90^\circ$.

Indeed, the cosine rule yields still more: it also implies the *inequality form of the PT*, drawing on the fact that the cosine of an angle is positive if the angle is acute, and negative if the angle is obtuse. Therefore, if $\angle A < 90^\circ$ then $a^2 < b^2 + c^2$, and if $\angle A > 90^\circ$ then $a^2 > b^2 + c^2$. So there is a lot of information contained in that simple rule.

So surely the cosine rule can be regarded as a genuine generalization of the PT?

Next we study the **theorem of Apollonius**. This is named after the third century AD Greek geometer Apollonius (often described as "the greatest geometer of antiquity", and known for his work on the conic sections), and it states the following:

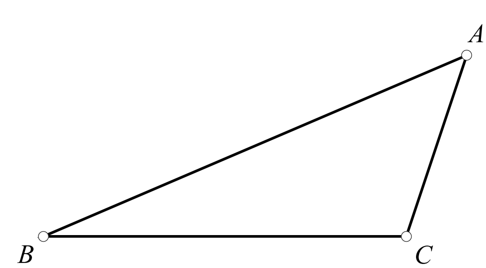
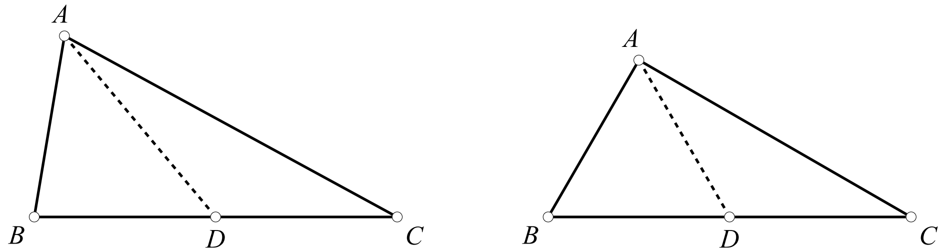


Figure 3. The cosine rule



(a): General case (b): The case when $\measuredangle A = 90^\circ$

Figure 4. The theorem of Apollonius

Theorem 2 (Apollonius). In $\triangle ABC$, let D be the midpoint of BC. Then:

$$AB^2 + AC^2 = 2\left(AD^2 + BD^2\right).$$

See Figure 4(a). In the special case when $\angle A = 90^\circ$ (see Figure 4 (b)), D is the centre of the circle through A, B, C, so $AD = BD$. The statement now reduces to: $AB^2 + AC^2 = 4BD^2$. Since $2BD = BC$, this may be written as $AB^2 + AC^2 = BC^2$. So this theorem too yields the PT as a special case.

With some ingenuity one can derive the inequality form of the PT from the theorem of Apollonius. (We urge you to try doing so.)

A symmetrical and more pleasing form of this theorem is the following: *The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.* This is equivalent to the theorem of Apollonius because of the easily-proved result that the diagonals of a parallelogram bisect one another.

Apollonius's theorem may itself be expressed in a stronger form, and we have the following theorem first found by a Scottish mathematician of the eighteenth century, Mathew Stewart (see Figure 5):

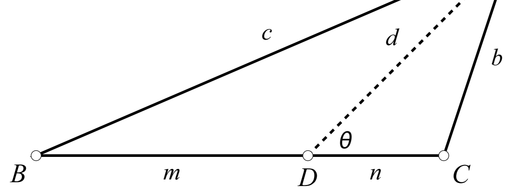


Figure 5. Stewart's theorem

Theorem 3 (Stewart). In $\triangle ABC$, let D be a point on side BC. Let a, b, c be the lengths of the sides of the triangle, and let d, m, n be the lengths of AD, BD, CD. Then:

$$b^2 m + c^2 n = a\left(d^2 + mn\right).$$

If D is the midpoint of BC then we have $m = n = a/2$, hence the statement reduces to $\left(b^2 + c^2\right) a/2 = a\left(d^2 + a^2/4\right)$, giving $b^2 + c^2 = 2d^2 + a^2/2$. This is equivalent to the theorem of Apollonius. (Do you see why?) Hence the theorem of Apollonius may be regarded as a specialization of Stewart's theorem. This implies that the theorem of Pythagoras may be considered a specialization of Stewart's theorem.

Here is a way of proving Stewart's theorem. Let $\angle ADC = \theta$; then $\angle ADB = 180^\circ - \theta$. Using the cosine rule together with the identity $\cos(180^\circ - \theta) = -\cos\theta$, we have:

$$b^2 = d^2 + n^2 - 2dn\cos\theta,$$

$$c^2 = d^2 + m^2 + 2dm\cos\theta.$$

If we multiply the first relation by m and the second one by n and then add them, the terms containing $\cos\theta$ are eliminated and we then get: $mb^2 + nc^2 = md^2 + mn^2 + nd^2 + nm^2$. Since $m + n = a$, this may be written in a more convenient way:

$$mb^2 + nc^2 = ad^2 + amn = a\left(d^2 + mn\right).$$

Those of you who are familiar with the math Olympiads will know that Stewart's theorem is part of the staple diet for all mathletes.

And now … a paradox!

We did not present the proofs of the cosine rule and the theorem of Apollonius, as most 11th standard mathematics textbooks give these proofs. But if we study these proofs, and the one given above for Stewart's theorem, we find a paradoxical situation. *Namely, these proofs are based on the Pythagorean theorem!* A good exercise for you would be to study these proofs and find the exact point(s) where the PT has been used. It may well come in a disguised form! For example, you may opt for a vector proof of the cosine rule as follows: Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ denote the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$ respectively (see Figure 3). Then $\mathbf{w} = \mathbf{u} - \mathbf{v}$, and by squaring we get:

$$\mathbf{w} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\mathbf{u} \cdot \mathbf{v}.$$

But $\mathbf{w} \cdot \mathbf{w} = a^2$, $\mathbf{u} \cdot \mathbf{u} = b^2$, $\mathbf{v} \cdot \mathbf{v} = c^2$ and $\mathbf{u} \cdot \mathbf{v} = bc \cos A$. Hence

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Similarly, for the theorem of Apollonius (Figure 4), let $\mathbf{u}$ and $\mathbf{v}$ denote the vectors $\overrightarrow{DB}$ and $\overrightarrow{DA}$ respectively; then $\overrightarrow{DC} = -\mathbf{u}$, so $\overrightarrow{AB} = \mathbf{u} - \mathbf{v}$, $\overrightarrow{AC} = -\mathbf{u} - \mathbf{v}$. Now we obtain expressions for AB^2, AC^2, AD^2, BD^2 and confirm that the stated claim is true.

But where has the PT been used in these two derivations? We leave this puzzle for you to crack.

So we have here a situation where a theorem appears to lead to its own generalization; how apparently paradoxical! What are we to make of it? In such a case, should we not regard the generalization as "only a corollary" and not a generalization at all?

Or should we say that the cosine rule can be regarded as a generalization of the PT only if we can find a proof for the rule that is not based on the PT?

We leave you to ponder the matter.

Exercises

(1) Show how the theorem of Apollonius implies the inequality form of the PT.

(2) If a pair of propositions P and Q have the property that P $\Longrightarrow$ Q and also Q $\Longrightarrow$ P, what word is appropriate to describe the relationship between P and Q?

(3) Find some nice specializations of the cosine rule. (See what you get if you put $\theta = 60°$ or $120°$.)

(4) Consider the following pair of propositions:

(P) $(1 + x)^2 = 1 + 2x + x^2$.

(Q) $(a + b)^2 = a^2 + 2ab + b^2$.

We certainly get the impression that P is a specialization of Q; for, by putting $a = 1$, $b = x$ in Q, we get P. So it would seem that Q is a generalization of P. But is this so? Consider what happens in P if we put $x = b/a$ in P. We get:

$$\left(1 + \frac{b}{a}\right)^2 = 1 + \frac{2b}{a} + \frac{b^2}{a^2},$$
$$\therefore\ (a + b)^2 = a^2 + 2ab + b^2.$$

So we have P $\Longrightarrow$ Q as well as Q $\Longrightarrow$ P. In the light of this, would you still say that Q is a generalization of P?

See [1] for more such examples. Chapter 2 in [2] has an illuminating discussion on the process of generalization.

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in the classroom

Angle Bisectors in a Quadrilateral

A Ramachandran

The bisectors of the interior angles of a quadrilateral are either all concurrent or meet pairwise at 4, 5 or 6 points, in any case forming a cyclic quadrilateral. The situation of exactly three bisectors being concurrent is not possible. See Figure 1 for a possible situation. The reader is invited to prove these as well as observations regarding some of the special cases mentioned below.

Start with the last observation. Assume that three angle bisectors in a quadrilateral are concurrent. Join the point of

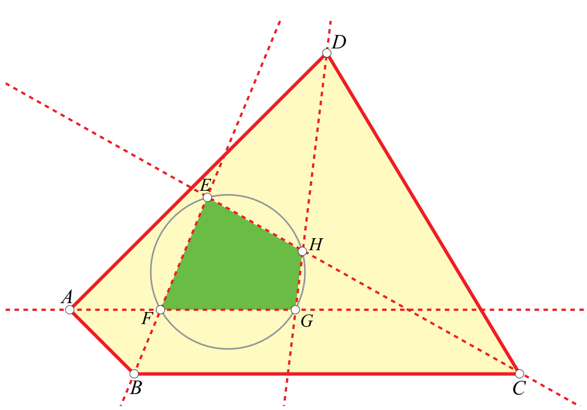


Figure 1. A typical configuration, showing how a cyclic quadrilateral $EFGH$ is formed

Keywords: *Quadrilateral, diagonal, angular bisector, tangential quadrilateral, kite, rhombus, square, isosceles trapezium, non-isosceles trapezium, cyclic, incircle*

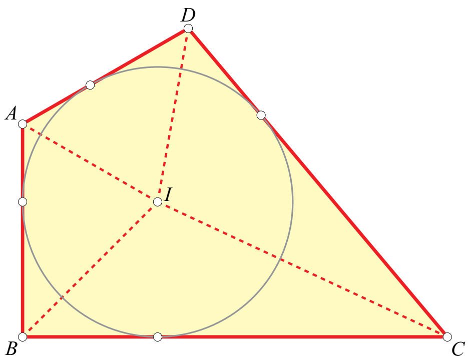


Figure 2. A tangential quadrilateral

concurrence to the fourth vertex. Prove that this line indeed bisects the angle at the fourth vertex.

A quadrilateral in which all the four angle bisectors meet at a point is a *Tangential quadrilateral* — one which has an circle touching all the four sides. This circle is the *incircle* of the quadrilateral, and its centre is the *incentre* of the quadrilateral (see Figure 2). Prove that in such a quadrilateral the sums of the lengths of opposite sides are equal (i.e., in Figure 2, $AB + CD = AD + BC$). Also prove the converse: If the sums of the lengths of opposite sides of a quadrilateral are equal, the quadrilateral is tangential.

Special cases of tangential quadrilaterals are the kite and rhombus (including the square). In the former, one diagonal bisects the angles at the vertices it joins, while this is true of both diagonals in a rhombus.

In the case of the bisectors meeting pairwise they form a quadrilateral. Prove that this is cyclic. Also prove the following:

(1) The angle bisectors in a general parallelogram form a rectangle (see Figure 3).

(2) The angle bisectors in a general rectangle form a square (see Figure 4).

(3) The angle bisectors in an isosceles trapezium (opposite side sums unequal) form a cyclic kite (also called a 'right kite'), where the equal angles are right angles (see Figure 5).

The case of a non-isosceles trapezium is shown in Figure 6. Here we get a quadrilateral in which one pair of opposite angles are right angles.

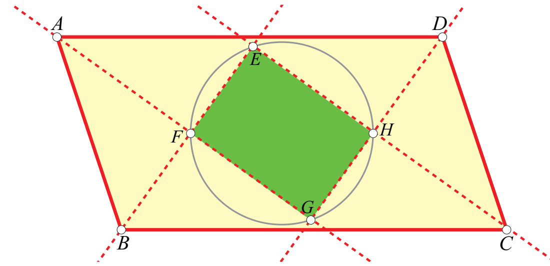


Figure 3. If $ABCD$ is a parallelogram, then $EFGH$ is a rectangle

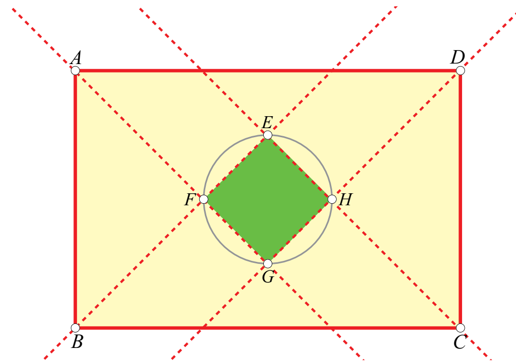


Figure 4. If $ABCD$ is a rectangle, then $EFGH$ is a square

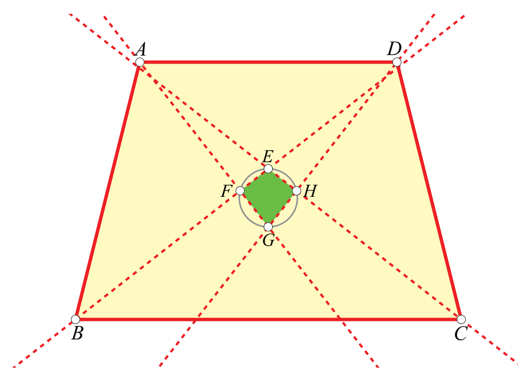


Figure 5. If $ABCD$ is an isosceles trapezium with opposite side sums unequal, then $EFGH$ is a cyclic kite

(4) In the case of a quadrilateral with only one set of opposite angles equal (opposite side sums unequal), the angle bisectors form an isosceles trapezium (see Figure 7).

Remark. We see from this survey that the simple act of drawing the internal bisectors of the four angles of a quadrilateral produces a configuration that offers unexpected riches. We invite the reader to supply proofs for the many claims made.

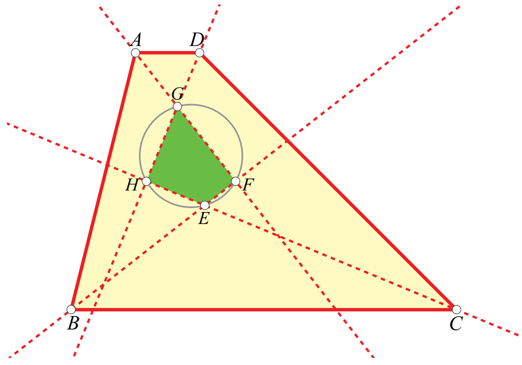


Figure 6. The case when $ABCD$ is a non-isosceles trapezium: the result is that $EFGH$ is a cyclic quadrilateral in which $\angle F = \angle H = 90°$

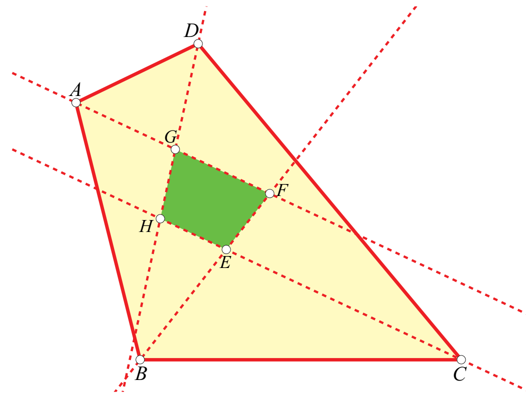
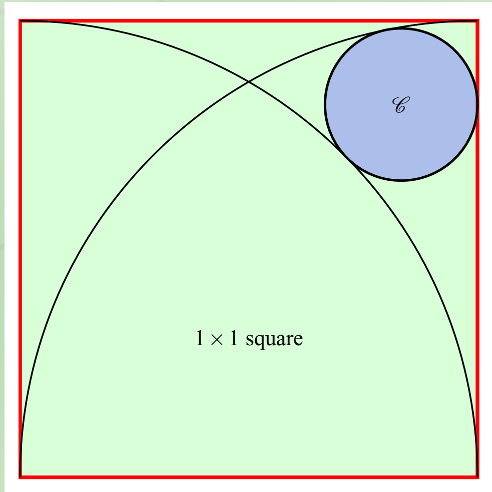


Figure 7. The case when $ABCD$ has $\angle B = \angle D$ but $\angle A \neq \angle C$: the result is that $EFGH$ is an isosceles trapezium ($FG \parallel EH$ and $EF = HG$)

A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

CHALLENGE



Can you find the radius of the inscribed circle $\mathscr{C}$?
The two quarter circles
(centred at the two bottom vertices of the square) have radius 1 unit each.

in the classroom

How to Prove It

This article continues the theme of offering multiple proofs of a single result, following entirely different themes and different starting points.

SHAILESH A SHIRALI

Introduction

In the last edition of this column we studied several cases of concurrence of lines in a triangle: (i) the perpendicular bisectors of the sides, (ii) the angle bisectors, (iii) the medians. Now we study the concurrence of the *altitudes* of a triangle. What is of some interest about this story is that there are so many ways to demonstrate concurrence, and they are so different from each other. We offer four different proofs of the result.

The context is this. Let triangle ABC be given. Draw the three altitudes of the triangle, i.e., through each vertex draw the line which is perpendicular to the opposite side. The claim then is that the three altitudes meet in a point known now as the *orthocentre* of the triangle. This point may lie within the triangle or outside it. See Figure 1 for two representative pictures.

First proof: Using Pythagoras theorem

This proof deserves to be better known than it is (because it shows that we can achieve the desired result with the use of the Pythagorean theorem and nothing more), so we start here. In Figure 2 we see a $\triangle ABC$ and its altitudes AD, BE, CF. We must show that they concur. We have deliberately drawn the altitudes *incorrectly*, so they appear *not* to concur. Let BE and CF meet at K, and let L be the foot of the perpendicular from K to BC. We must show that L coincides with D, for this implies that AD passes through K, and hence that the altitudes concur.

Keywords: *Altitudes, orthocentre, concurrence, cyclic, quadrilateral, vectors*

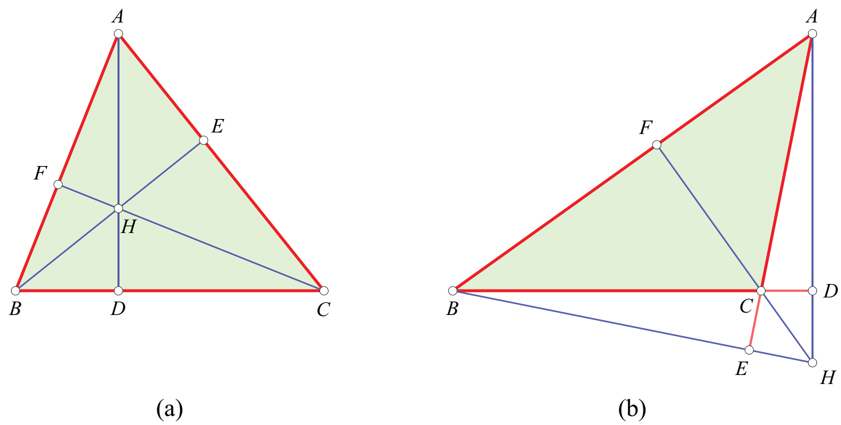


Figure 1. Concurrence of the altitudes of a triangle: Two possible configurations

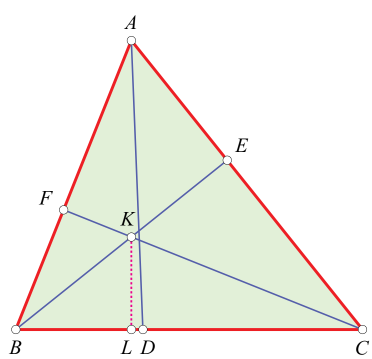


Figure 2. Proof using the Pythagorean theorem

Using the Pythagorean theorem repeatedly, we get the following relations:

$$\begin{cases} BL^2 - CL^2 &= (BL^2 + KL^2) - (CL^2 + KL^2) \\ &= KB^2 - KC^2, \\ CE^2 - AE^2 &= (CE^2 + KE^2) - (AE^2 + KE^2) \\ &= KC^2 - KA^2, \\ AF^2 - BF^2 &= (AF^2 + KF^2) - (BF^2 + KF^2) \\ &= KA^2 - KB^2. \end{cases}$$

From this it follows, by addition, that

$$BL^2 - CL^2 + CE^2 - AE^2 + AF^2 - BF^2 = 0. \quad (1)$$

We also have:

$$\begin{cases} BD^2 - CD^2 &= (BD^2 + AD^2) - (CD^2 + AD^2) \\ &= AB^2 - AC^2, \\ CE^2 - AE^2 &= (CE^2 + BE^2) - (AE^2 + BE^2) \\ &= BC^2 - BA^2, \\ AF^2 - BF^2 &= (AF^2 + CF^2) - (BF^2 + CF^2) \\ &= CA^2 - CB^2. \end{cases}$$

From this it follows, by addition, that

$$BD^2 - CD^2 + CE^2 - AE^2 + AF^2 - BF^2 = 0. \quad (2)$$

Comparing (1) and (2) we see that $BL^2 - CL^2 = BD^2 - CD^2$, and hence that

$$(BL - CL) \cdot (BL + CL) = (BD - DC) \cdot (BD + DC),$$

$$\therefore\ (BL - CL) \cdot BC = (BD - DC) \cdot BC,$$

$$\therefore\ BL - CL = BD - DC$$

(since BC is clearly not zero).

Hence $BL - BD = CL - DC$, or $LD = -LD$, giving $LD = 0$. Hence L and D are the same point. The desired end has been reached. Note that the proof has been written assuming that the points L and D lie on segment BC and not on its extension, but with small modifications (which we leave for you to do) it will hold for the other possibilities as well.

Second proof: Using circle theorems

In Figure 2 we see the altitudes BE and CF of $\triangle ABC$ intersecting at point H. The line through A and H meets line BC at D. *Note that AD is not assumed at the outset to be an altitude.* (That is why AD has been drawn in a different way from BE and CF, and the relevant angles at E and F are marked as right angles, but not the angle at D.) Rather, we must *show* that AD is an altitude. Here is how we reason it out.

- Quadrilateral $AEHF$ is cyclic because $\angle AEH$ and $\angle AHF$ are right angles. Hence $\angle AHE = \angle AFE$ ("angles in the same segment of a circle").
- Quadrilateral $BFEC$ is cyclic because $\angle BFC$ and $\angle BEC$ are right angles. Hence $\angle AFE = \angle ECB$ ("exterior angle of a cyclic quadrilateral equals the interior opposite angle").

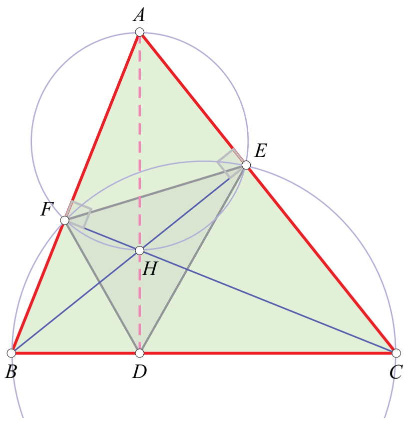


Figure 3. Proof using circle theorems

- Hence $\angle AHE = \angle ECB$. This implies that quadrilateral $DCEH$ is cyclic.
- Therefore $\angle HDC + \angle HEC = 180°$. But $\angle HEC = 90°$. Hence $\angle HDC = 90°$. That is, HD is perpendicular to BC. This the same thing as saying that $AD \perp BC$.

Third proof: Using coordinates

In adopting a coordinate-based approach, the first thing is to choose the axes wisely. In Figure 4 we see $\triangle ABC$ for which the side BC lies along the x-axis and the altitude AD lies along the y-axis. (The wisdom of this choice of axes will become apparent shortly.) Our aim will be accomplished by showing that the perpendiculars from B to line AC and from C to line AB meet on the y-axis.

Assign coordinates as shown at the right of Figure 4: $A = (0, a)$, $B = (b, 0)$, $C = (c, 0)$; here of course $b \neq c$, since B and C are distinct points. We first find the equation of the perpendicular from B to line AC. The slope of AC is $-a/c$, so the slope of the perpendicular is c/a, and the equation of the perpendicular is $y - 0 = (c/a)(x - b)$, i.e., $ay = c(x - b)$. This may be expressed as follows:

$$cx - ay = bc. \tag{3}$$

By symmetry we may deduce the equation of the perpendicular from C to line AB simply by switching the roles of b and c in the above equation. Here is what we get:

$$bx - ay = bc. \tag{4}$$

To find where the two altitudes intersect, we solve the above two equations for x and y (in fact, it is enough if we solve just for x). By subtraction we get

$$(b - c)x = 0, \quad \therefore\ x = 0. \tag{5}$$

Hence the two altitudes intersect on the y-axis, just as we wanted. It follows that the three altitudes concur.

Fourth proof: Using vectors

The last proof we offer uses vectors. Let altitudes BE and CF meet at H as in Figure 5. We wish to prove that AH is perpendicular to BC. Take H to be the origin relative to which position vectors are referred, and let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C respectively. The fact that

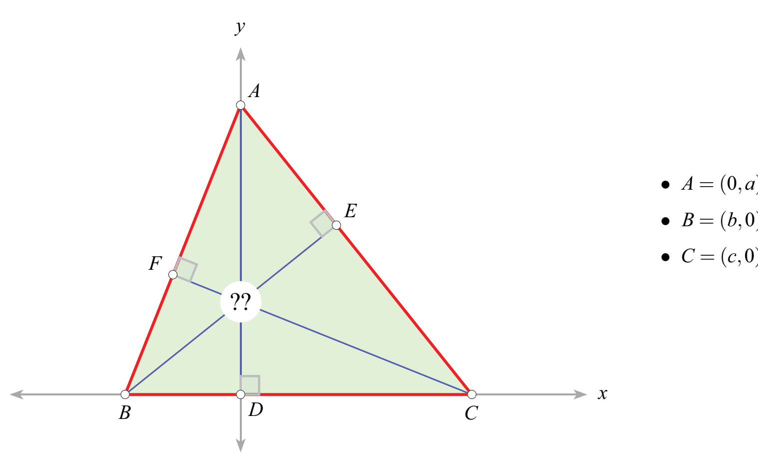


Figure 4. Proof using coordinates. The '??' indicates that we do not know whether or not the lines concur.

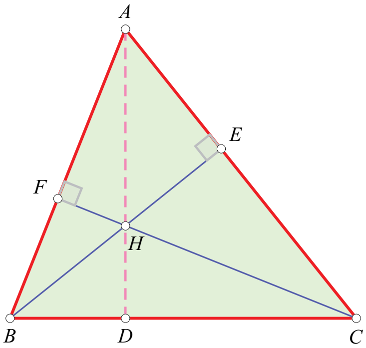


Figure 5. Last but not least: a vector proof

$BH \perp AC$ and $CH \perp AB$ then expresses itself in the following statements:

$$\begin{cases} \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) &= 0, \\ \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) &= 0. \end{cases} \tag{6}$$

The first of these yields

$$\mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b},$$

while the second one yields

$$\mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}.$$

Hence we have: $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, that is:

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0.$$

This is the same thing as saying that $AH \perp BC$.

Here is a query for you to ponder: Is the vector proof really that different from the proof based on coordinates?

Closing comments

So there you have it: four entirely different proofs for a single result. You may find it instructive to compare them and contrast their distinctive features.

Does this compilation exhaust the list of proofs available for this result? Not at all! We can think of at least two more proofs. One proceeds by constructing a triangle with twice the dimensions of the given triangle and invoking the midpoint theorem and the properties of a parallelogram; this may well be the most elegant proof available. Yet another proof uses the idea of the *radical axis* of two circles. We leave you to explore these for yourself.

In closing we point out the following interesting and attractive result. For any triangle ABC, let H denote its orthocentre. Consider the set of vertices $\{A, B, C, H\}$. This set has the following property: *For each subset of three points from the set, form a triangle with those points as its vertices; then the fourth point is the orthocentre of this triangle.* In other words, if H is the orthocentre of $\triangle ABC$, then A is the orthocentre of $\triangle HBC$; B is the orthocentre of $\triangle HCA$; and C is the orthocentre of $\triangle HAB$. For this reason, the set $\{A, B, C, H\}$ is called an *orthocentric set*.

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in the classroom

Low floor high ceiling tasks

Investigations with Pentominoes

Hit the ground running!

Swati Sircar & Sneha Titus

Beginning with this issue, we start a new series which is a compilation of 'Low Floor High Ceiling' activities. What does this term mean? I first encountered it at a Cambridge workshop by Charlie Gilderdale and the concept appealed instantly. The name describes it perfectly: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that each student is pushed to his or her maximum as they attempt their work. There is enough work for all but as the level gets higher, fewer students are able to complete the tasks. The point however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

I was recently asked how I handled differentiated teaching in the classroom. The very act of reaching out to academically weak students seems to defeat the purpose as being singled out for such attention sends their already shaky confidence nose-diving to new lows. After many such attempts, I realised that my efforts needed to be much more subtle and should have as a primary objective ways in which to increase the student's confidence. I found 'Low Floor High Ceiling' tasks a great way to deliver differentiated instruction subtly. Since the task starts off fairly simply, the initial levels can be attempted successfully by all the students in the class. The more able students sail through these levels but the greater advantage is that even the less able students are able to get a start on the problem. This boosts their confidence and sustains their interest. As the task proceeds, the challenge level increases. But the earlier steps provide the scaffolding that can help students attempt these levels.

There is also opportunity for students to make conjectures based on observations and even prove these conjectures if they are motivated enough.

Teachers may find the development of such tasks rather challenging but a good collection will prove invaluable. Jo Boaler's youCubed has started putting together a series of such math tasks which you can access at http://youcubed.stanford.edu/tasks/.

We hope to augment this collection with a new activity in each issue. We have tried to build mathematical skills in the activity, designing it so that students can observe, make conjectures and even prove these if they are motivated enough. We start with the Pentomino Kit which was featured in the July 2014 issue. If you haven't done so already, we strongly advise you to read this article (also available on http://www.teachersofindia.org/en/article/pentominoes).

Each card (or set of cards) is a task which features a series of questions which build up in complexity.

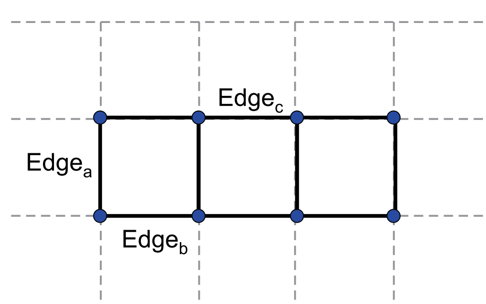
To get you started, think of a monomino as a unit square.

TASK 1

Let's start at the very beginning…

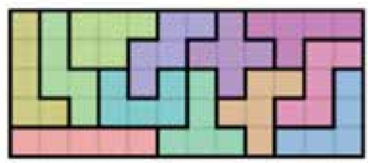
- What is a monomino? Show how many there can be.
- What is a domino? Show how many dominoes there can be.
- What is a tromino? Show how many trominoes there can be.
- What is a tetromino? Show how many tetrominoes there can be.
- Is there a way to identify unique edges of an n-omino to which unit squares can be attached?
- Using this, can you count how many n+1-ominoes will be formed?
- So how many pentominoes are there?

Teacher's Note: Some edges of this tromino have been labelled. Attaching a unit square to any one of these edges will give you a different tetromino. A systematic labelling or colouring of the edges of an n-omino will help students arrive at all possible (n+1)-ominoes. Students can also draw a tree diagram of this process. Interestingly, the branches of the tree will start intersecting since different n-ominoes can generate the same (n+1)-omino!



At this point if you want to make a pentomino kit, here is a quick and easy method.

1. Take a sheet of thick cardboard and cut a 5 inch by 12 inch piece
2. Make a 5 x 12 grid of 1 inch squares on it.
3. Use the following template for the pentominoes kit



5 × 12

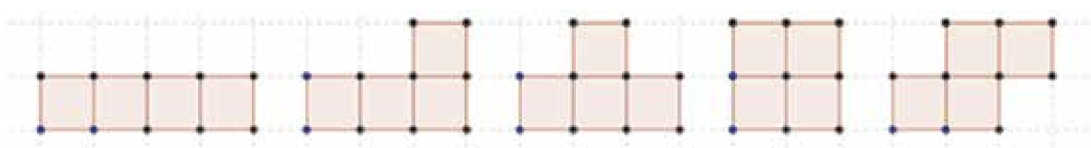
4. Cut out the shapes.
5. Colour each shape.

TASK 2

Now that you have the shapes in hand, you can study various features of each pentomino and classify them.

Classification based on convex and non-convex

- Which of the pentominoes are convex?
- Which are non-convex?
- How are the factors of 5 related to this result?
- Can you confirm this finding with the tetrominoes?



Teacher's Note: The convex n-ominoes will be rectangles. Since 5 is prime, the only possible rectangle is the 5x1 rectangle. Students can note that there will be more than one convex tetromino since 4 is not a prime.

TASK 3

Classification based on symmetry

- Which pentominoes need to be coloured only on one side? Why?
- How many lines of symmetry do each of the "one-sided" pentominoes have?
- Look closely at those with more than one line of symmetry.

Do they have any other kind of symmetry as well? What symmetry is that, if any?

- Do any of the "double sided" pieces have rotational symmetry?
- Comment on the order of rotational symmetry and the number of lines of symmetry for each of these pentominoes.

Teacher's Note: Any shape with more than one line of symmetry must have rotational symmetry as well. And the converse, any shape with both rotational and line symmetry must have at least 2 lines of symmetry. Both can be proved quite simply. There is a relation: minimum angle between 2 lines of symmetry = ½ × minimum angle of rotation. Students may find it helpful to fill in the following table and then conjecture on the relationship between linear and rotational symmetry.

		Linear symmetry	
		Yes	**No**
Rotational symmetry	**Yes**		
	No		

TASK 4

Name of Piece	Perimeter of piece	Number of lines separating squares in each piece
F		
I		
L		
N		
P		
T		
U		
V		
W		
X		
Y		
Z		

Any connection between column two and column three?

Teacher's Note: Pattern recognition while doing this task can lead students to the formula for the perimeter of a pentomino to be $20 - 2n$, where n is the number of lines separating squares in each piece.

TASK 5

Enveloping rectangles

- Why are the perimeters different for the pentominoes enclosable in 2 × 3 rectangles?
- Is there any tetromino whose perimeter equals that of a pentomino? Note: area of any tetromino is 4 whereas that of any pentomino is 5.
- Look at all trominoes, tetrominoes and pentominoes:
- Any other possible pairs with same perimeter but different area?
- Why is the perimeter preserved?
- Any other possible pairs with same area but different perimeters? Why are the perimeters different?

Teacher's Note: Note that carving out an L from a corner does not change the perimeter, even when repeatedly performed. However, carving out a U increases the perimeter by 2 units.

TASK 6

- How many quadrilaterals?
- How many hexagons?
- What other polygons do you get? Tabulate what you found as follows:

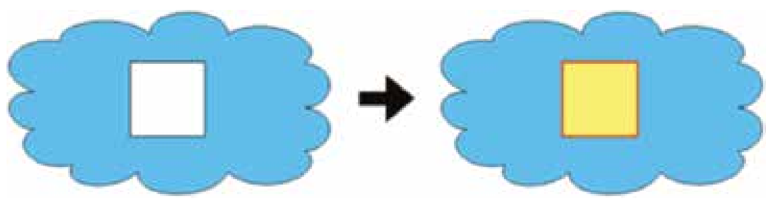
Number of sides	Which pentomino(es)	Total number
4	*I*	*1*
6		

- Are there any n-ominoes with an odd number of sides?

Teacher's Note: The proof that all n-ominoes have an even number of sides is an interesting illustration of the use of proof by induction. We provide it below:

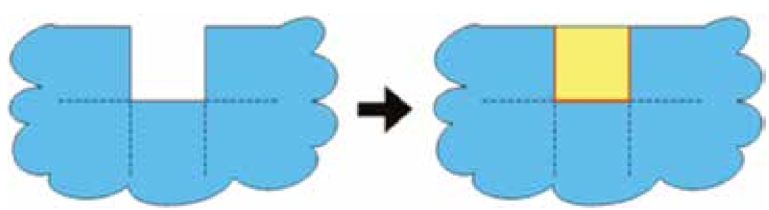
Proof by Induction

- For n = 1: monomino is a square which has an even number of sides
- Assume that for all n = 1, …, m, all possible n-ominoes have even numbers of sides
- We will show that when we create an (m + 1)-omino from m-omino by adding a square, the parity of the (m + 1)-omino and the m-omino does not change. In fact, the number of sides remains unchanged or changes by 2 or 4 only.
- Going from m-omino to (m + 1)-omino. We consider the different possibilities for the number of edges in common between the existing m-omino and the added square. This can be 1, 2, 3 or 4. We examine each case in turn. Note that in the case of contact with 2 edges, the two edges could be either adjacent to each other, or opposite to each other.
- Case A: 4 sides in common. This happens when there is a hole/island in the interior of the m-omino and the new square is inserted into the hole.



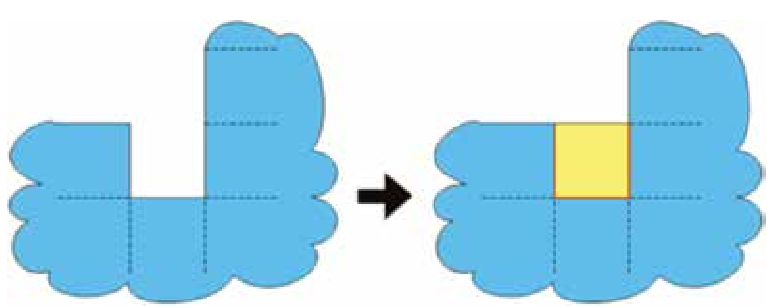
As the figure shows, the number of sides goes down by 4 and thus preserves its parity.

- Case B: 3 sides in common. Several subcases must be distinguished here, depending on the nature of the space in the m-omino where the new square has been attached. The middle side where contact is made must be 1 unit long, but the other two sides could be of different lengths. Both may be 1 unit long, or one may be 1 unit and the other may be more than 1 unit, or both may be more than 1 unit long.



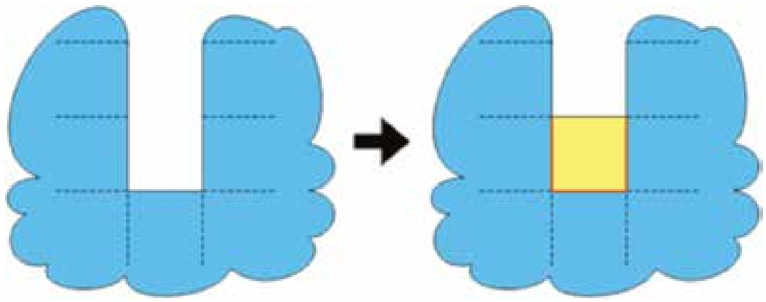
Case B.1: Both sides 1 unit in length: p-gon → (p – 4)-gon

As can be seen, the number of sides goes down by 4 and this preserves its parity.



Case B.2: One side 1 unit, other side > 1 unit: p-gon → (p – 2)-gon

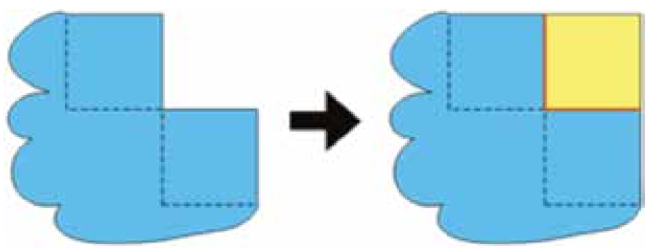
The number of sides goes down by 2 and thus preserves its parity.



Case B.3: Both sides > 1 unit: p-gon → p-gon

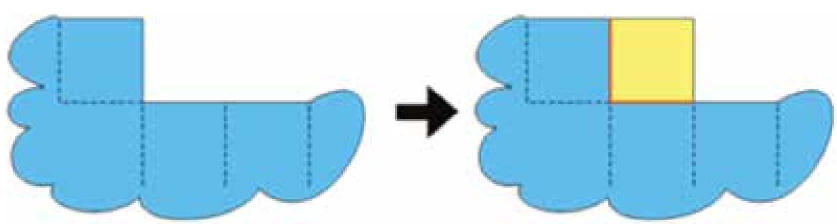
Here the number of sides remains the same and thus preserves its parity.

- Case C: 2 sides in common. As earlier, several subcases arise, depending on the lengths of the two sides where the new square has been attached. The two sides may both be 1 unit long, or one may be 1 unit and the other may be more than 1 unit, or both may be more than 1 unit long. Also, the two sides may be adjacent to each other or opposite each other.



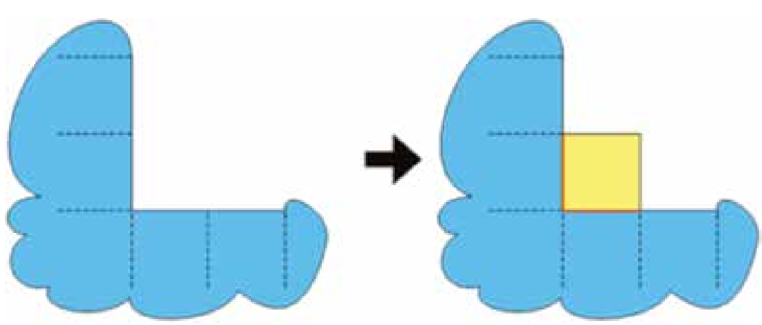
Case C.1: Adjacent sides, both 1 unit: p-gon → (p – 2)-gon

The number of sides goes down by 2 and thus preserves its parity.



Case C.2: Adjacent sides, one side 1 unit, other side > 1 unit: p-gon → p-gon

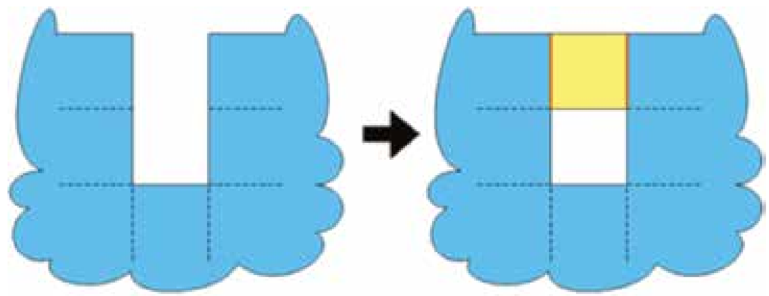
The number of sides remains the same and thus preserves its parity.



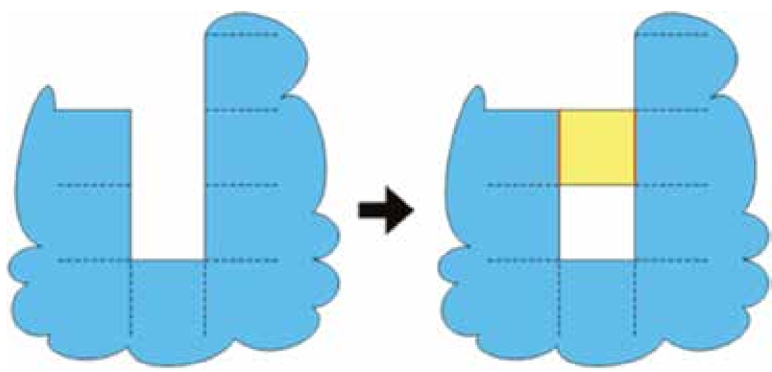
Case C.3: Adjacent sides, both > 1 unit: p-gon → (p + 2)-gon

The number of sides goes up by 2 and thus preserves its parity.

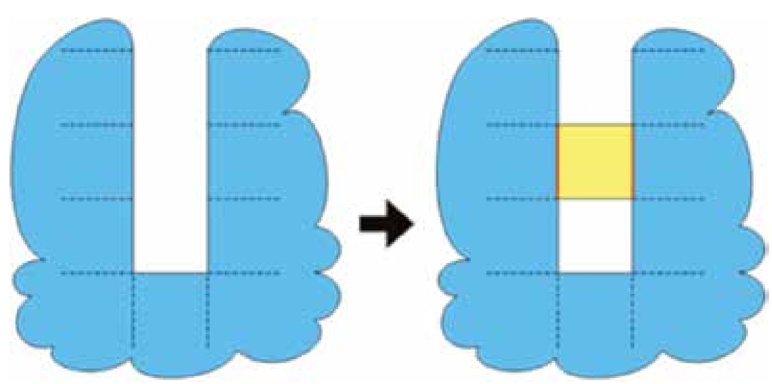
Next, consider the cases where the edges of contact are opposite edges of the square. The same subcases arise: the two sides may both be 2 unit long, or one may be 2 unit and the other may be more than 2 unit, or both may be more than 2 unit long.



Case C.4: Opposite sides with length 2 unit each: p-gon → p-gon



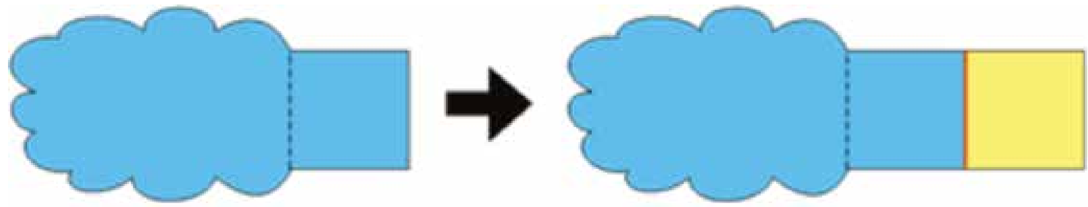
Case C.5: Opposite edges with lengths 2 unit and > 2 unit: p-gon → (p + 2)-gon



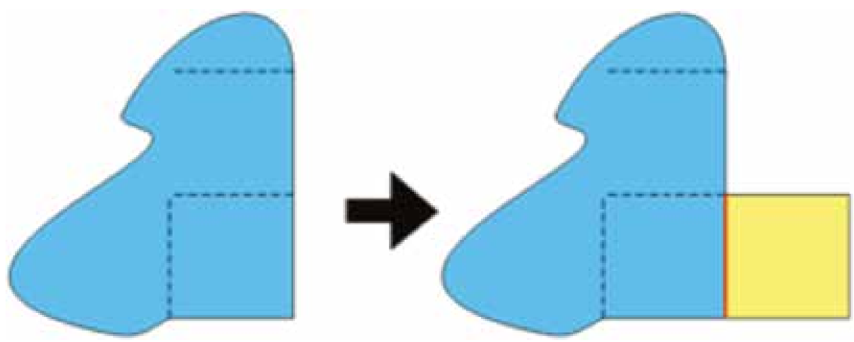
Case C.6: Opposite edges both with length > 2 unit: p-gon → (p + 4)-gon

In each case the number of sides preserves its parity, as it changes by 0, 2 or 4.

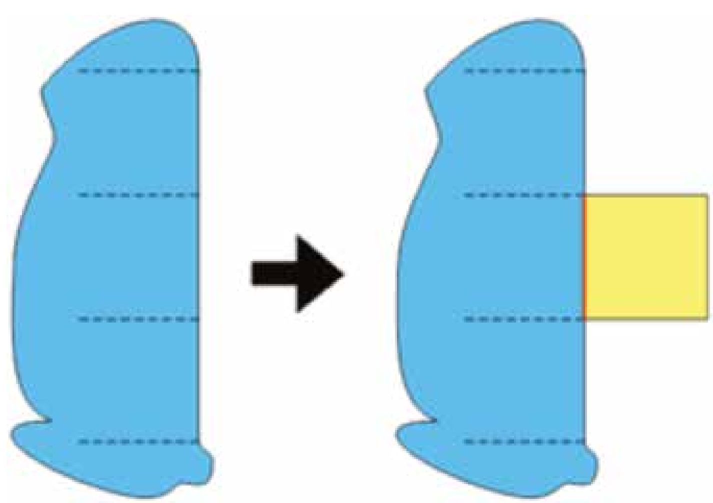
- Case D: 1 side in common. Once again, several subcases arise, depending on the length of the side where the new square has been attached. The length may be 1 (if the new square has been attached at an end) or > 1 (if it is attached somewhere in the middle). As earlier the pictures shown below settle the issue.



Case D.1: p-gon → p-gon



Case D.2: p-gon → (p + 2)-gon



Case D.3: p-gon → (p + 4)-gon

This enumeration of cases proves the claim: the number of sides always changes by an even number (0, 2 or 4). As it is even at the start, it always remains even.

Closing Comments

Through this Low Floor High Ceiling activity we have tried to illustrate with the pentominoes (and polyominoes in general) how activities involving something basic (for example, counting) can build up to richer mathematics, through finding common patterns, hypothesizing and validating the same:

- Understanding area and perimeter and how they change with respect to each other (Tasks 4, 5);
- Identifying a variable and formulating an expression with it to calculate another variable (Task 4) – this is where a student beginning algebra can play with variables (independent and dependent) and create algebraic expressions and equations using them;
- Observing a pattern, formulating a hypothesis and proving it using induction (Task 6) – a powerful tool which students rarely get to see before the higher secondary stage.

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techspace

Learning Mathematics through Geometrical Inquiry

The possibility of learning mathematics through ICT support has never before been so promising. With the free software GeoGebra on your computer, you can conduct mathematical investigations in a time frame that people some generations back never could have dreamed of. Since the GeoGebra environment allows you to do calculations, geometrical constructions, measurements, and modelling in the same file, you can explore mathematics in a wonderful way. I am particularly fond of geometrical inquiry based problems that can open up other areas of mathematics.

Thomas Lingefjärd

Introduction

Traditionally, the use of tools such as compass, protractor and ruler has been a common practice in learning geometry in school classrooms. However, in recent years, a class of software tools known as Dynamic Geometry Software (DGS), has revolutionised the teaching and learning of geometry and has brought about a shift of paradigm in the way concepts in geometry can be made accessible to students.

Some of the DGS popularly used in schools across the world include GeoGebra (Hohenwater, 2001), CabriTM (Laborde and Bellemain, 1993) and Geometer's SketchpadTM (Jacjiw, 1991). One of the greatest affordances of a Dynamic Geometry Environment (DGE) is that it allows the user to drag parts of a geometrical figure in a geometry window, while the measurements of the figure change dynamically in an algebra window. Thus geometrical figures are dynamic and not static as they are when drawn on paper. The dynamic nature of a DGE figure enables the user to observe properties about it which

Keywords: *Algebra, Geometry, Modelling, Investigations, Reasoning*

remain invariant and those which do not. Since a DGE also provides measurements of different parts of the figure, such as the lengths of sides and angle measures, it enables the user to make conjectures by observing the changes or invariance in the measurements. In fact, in a DGE, the underlying principle is "to provide a family of diagrams as representing a set of geometrical objects and relations instead of a single static diagram". Researchers have also described a DGE as a 'microworld' which provides rich opportunities for students to make and test conjectures. Thus a DGE provides a environment for performing investigations and for working on inquiry based learning activities and the mathematics education community has strongly emphasised the same (Brown &Walter, 2005; Da Ponte, 2007; Jones & Shaw, 1988; Leikin, 2004;Silver, 1994; Wells, 1999, 2001).

In this article we shall describe the exploration of a theorem in geometry which can provide opportunities for performing investigations, observing patterns and making conjectures. The DGS used as the vehicle for exploration is GeoGebra. It is free and open-source and may be downloaded from www.geogebra.org.

Personally, my view of geometrical investigations changed over twenty years ago, when I first read about Walter's theorem and the investigations done by a grade 9 student, Ryan Morgan. Walter's theorem was first presented as Marion Walter's theorem in the November 1993 issue of the *Mathematics Teacher* in the section called Readers Reflections (Cuoco, Goldenberg, and Mark 1993). In short it says the following, illustrated by figure 1.

Marion Walter's theorem appeared again in the May 1996 issue of the *Mathematics Teacher* in an article called Morgan's theorem (Watanabe, Hanson, and Nowosielski 1996). The article tells the story about young Ryan Morgan, a ninth grader with good mathematical sense and a strong desire to explore a problem to its limits. Ryan's mathematics teacher Frank Nowosielski presented Walter's theorem to his class in the fall of 1993 and asked them to explore if it would hold for various types of triangles. Ryan was not satisfied merely with verifying Walter's theorem. He was interested in finding out what would

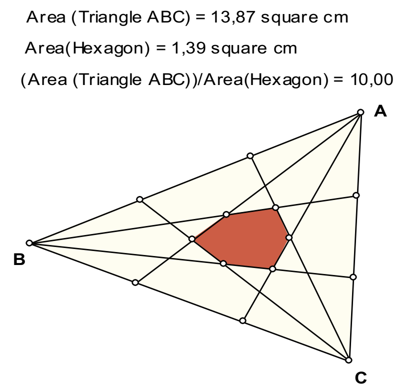

Figure 1. Illustrating Walter's theorem: If the points of trisection of the sides of a triangle are joined to the opposite vertices, the hexagon formed inside the triangle has one-tenth the area of the triangle

happen if the sides of the triangles were divided into more than three congruent segments. Ryan and his teacher called the process of dividing a side of a triangle into n congruent segments "n-secting". Using the Geometer's Sketchpad, Ryan experimented with different n-sections (Watanabe et. al., p. 420). The article explains the methodology of Ryan's investigations and what it led to; not surprisingly, Ryan was invited to present his conjecture at a special mathematics colloquium at Towson State University in 1994. If you are interested in how pre-service mathematics teachers handled the challenges of this problem, read about it in Lingefjard & Holmquist (2003).

A configuration similar to Walter's Theorem,but not equivalent to it, comes from the following exploration. Observe the two equilateral triangles in Figure 2. The side of the larger triangle has been trisected in order to construct the inner triangle, which is formed when we join each vertex of the triangle to one of the trisecting vertices on the opposite side, in a cyclically symmetric way. One finds now that the ratio of the areas of the two triangles is exactly 1 : 7. The reader may verify this using GeoGebra.

Investigation 1

Measure the same ratio if you divide the side of the outer triangle into 5 equal sections and construct the smallest possible equilateral triangle inside. (This happens if you join each vertex of the original triangle to the vertex on the opposite side

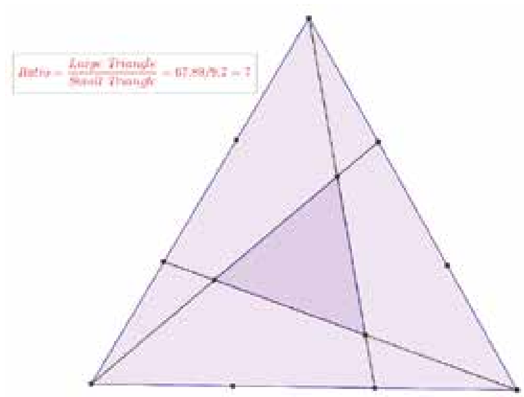

Figure 2. GeoGebra figure showing that the ratio of areas of the inner and outer triangles is 1:7

which is closest to the midpoint of that side, in a cyclically symmetric way.) Repeat the procedure for 7, 9, 11 and 13 equal sections. Do you see any pattern emerging in the sequence of ratios?

The activity is easy to start with, but soon presents an interesting challenge. Firstly, we need a method for dividing a line segment into n equal parts, a process which we refer to as n-secting. This is easy to accomplish using the available tools of GeoGebra (which are more powerful than the tools of the Geometry instrument box). The code for finding the ratio of areas is given at the end of the article.

Figure 3 (a) and 3 (b) show the output in GeoGebra of dividing a side of the equilateral triangle into 5 equal parts and 7 equal parts respectively.

Investigation 2

Do you see any pattern in the sequence of ratios of the areas of the inner and outer triangles in the various cases of 'n-secting' the sides of the outer triangle? From the cases of $n = 3$, 5 and 7, can you make a conjecture regarding the ratio for $n = 9$ or 11?

From our constructions we find the following. For a 3-section the ratio of the areas is $1 : 7$. For a 5-section and a 7-section the ratios are $1 : 19$ and $1 : 37$ respectively. We may also say that the ratio for a 1-section is $1 : 1$ (this is true in a rather trivial sense). The numbers 1, 7, 19, 37 form a sequence in which the differences between the first three pairs of consecutive numbers are 6, 12, 18. Can we conclude that the next increment will be 24 and

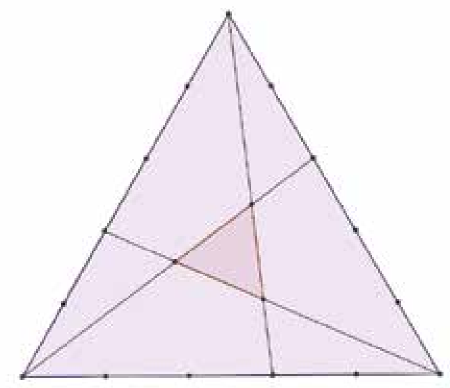

Figure 3(a)

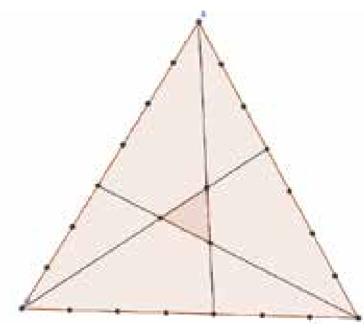

Figure 3(b)

Figure 3. GeoGebra figures showing the inner equilateral triangles formed after (a) 5-secting, (b) 7-secting the sides of the outer equilateral triangle. The reader may compute the ratio of the areas of the inner and outer triangles using GeoGebra.

the ratio will be $1 : 61$ (since $37 + 24 = 61$)? Let us find out. See Figure 4. A GeoGebra investigation shows that the ratio is indeed $1 : 61$ for the case $n = 9$. Thus we have experimentally verified our conjecture for the next term.

The ratios lead us to this sequence: 1, 7, 19, 37, 61. An attempt to find a pattern in this sequence reveals the following:

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 7 = 1 + 1 \cdot 6, \\ a_2 &= 19 = 1 + 2 \cdot 6 + 1 \cdot 6, \\ a_3 &= 37 = 1 + 3 \cdot 6 + 2 \cdot 6 + 1 \cdot 6, \\ a_4 &= 61 = 1 + 4 \cdot 6 + 3 \cdot 6 + 2 \cdot 6 + 1 \cdot 6. \end{aligned}$$

Generalising this pattern for the k-th term we get

$$a_k = 1 + k \cdot 6 + (k-1) \cdot 6 + (k-2) \cdot 6 + \ldots + 2 \cdot 6 + 1 \cdot 6.$$

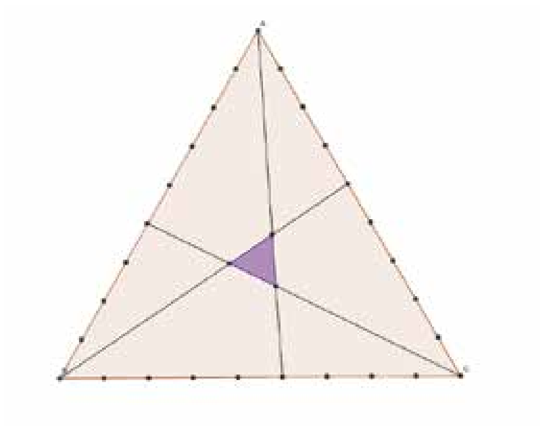

Figure 4. GeoGebra finds the ratio of the areas of the inner and outer triangle to be 1:61

Using the formula for the sum $1 + 2 + 3 + \ldots + k$, we get:

$$a_k = 1 + 6\frac{k(k+1)}{2} = 3k^2 + 3k + 1.$$

The observed pattern therefore suggests that the ratio between the areas of the triangles is governed by the formula $a_k = 3k^2 + 3k + 1$. But note here that the k in this formula is **not** the same as n, the number of parts into which the sides of the triangle were divided. The values of n corresponding to $k = 1, 2, 3, 4, \ldots$ are $n = 3, 5, 7, 9, \ldots$ from which we see that $n = 2k + 1$. This yields $k = (n-1)/2$. Substituting for k into the above relation, we find that the formula for the ratio of areas in terms of n is $(3n^2 + 1)/4$. We may check that by substituting $n = 3, 5, 7, 9$ into this formula, we get the numbers $7, 19, 37, 61$.

But maybe there is more to find out about the numbers $1, 7, 19, 37, 61, \ldots$?

A web search for the numbers $1, 7, 19, 37, 61, \ldots$ reveals that they are called ***hexagonal centred numbers*** and they are obtained from counting the number of spots making up a full hexagon.

Let us now explore the sequence $1, 7, 19, 37,$ …graphically, by creating ordered pairs of points $(n, t(n))$, where n represents the number of the term and $t(n)$ the n-th term of the sequence. The first three points are:$(0, 1), (1, 7), (2, 19)$. Three points not in a line are enough to fix a quadratic curve. When we fit a quadratic function through the points we get a graph shown in Figure 6. Note that the function $f(x) = 3x^2 + 3x + 1$ is the same as the formula obtained by us when we generalised our sequence algebraically.

Now when we have this model, we can go back to Figure 2 and make some more observations.

If we label the length of the sides of the outer triangle as s and the length of the sides in the inner triangle as x, then the distance from one of the inner triangles' vertex to the closest vertex of the large triangle is kx, where k is a positive number whose value has to be found. Now the segments AP, CQ, BR have equal length. In triangle AOC, $AC = s, OC = kx, AO = (k+1)x$ and angle AOC = 120 degrees (which is so because angle AOR equals 60 degrees). Hence by the cosine rule

$$AC^2 = AO^2 + OC^2 - 2AO.OC.\cos AOC.$$

This leads to:
$s^2 = (k+1)^2 x^2 + k^2x^2 + 2k(k+1)x^2/2$, i.e.,

$$\left(\frac{s}{x}\right)^2 = 3k^2 + 3k + 1.$$

This equation has a link to the formula GeoGebra helped us find. Note that k indicates a specific ratio between the two triangles but it is **not** the same as n, the number of parts into which we divided the sides of the outer triangle. The task of

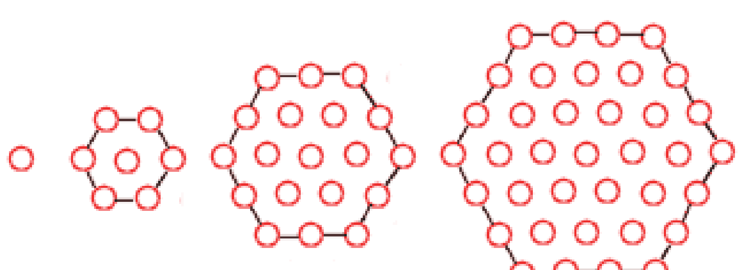

Figure 5. Hexagonal centred numbers (see http://www.drking.org.uk/hexagons/misc/numbers.html)

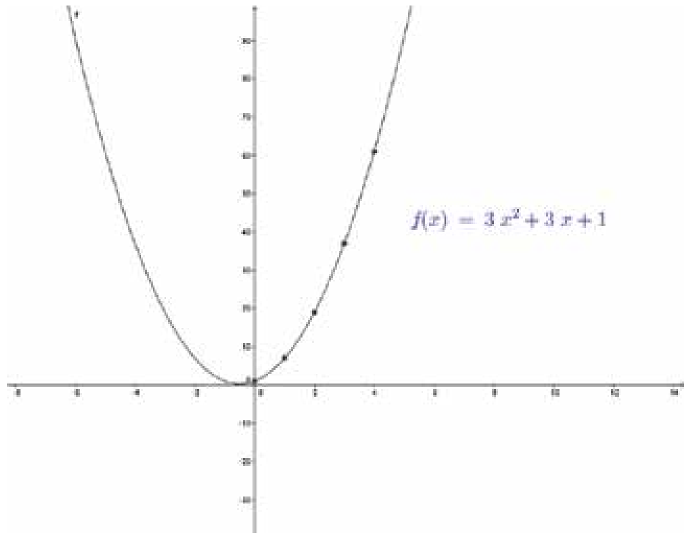

Figure 6. Modelling tools within GeoGebra

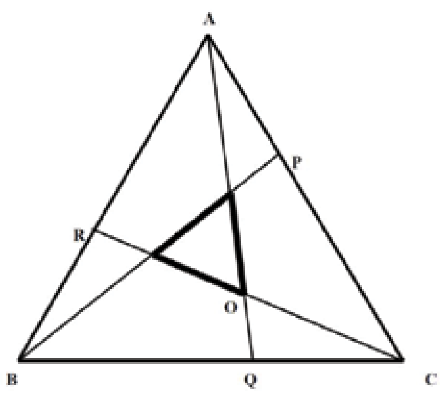

Figure 7. Using the cosine rule to compute some ratios

theoretically proving the formula $(3n^2 + 1)/4$ for the ratio of the areas of the triangles remains to be done, but we leave this for the reader.

This exploratory activity presents us with an opportunity to explore a problem both geometrically and algebraically. Pictorial representations on the computer screen (using GeoGebra) along with numerical and graphical representations of the sequence of ratios of areas of the equilateral triangles provide an opportunity to generalise the problem and create a link between the geometrical exploration and algebraic formulation.

Closing remarks

Some mathematics teachers see the potential of using dynamic geometry to explore mathematical concepts and to provide inquiry-based learning experiences to students. Others cite improvements in the classroom atmosphere, increase in motivation levels of students and the efficiency of showing many examples at once as some of the reasons for incorporating dynamic geometry in their classrooms (Lampert1993; Ruthven, et al. 2005).

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Appendix: Procedure for carrying out the investigation using GeoGebra

We give the steps in pseudo-code. The GeoGebra macro can be prepared using this code.

1. Select three non-collinear points A, B, C. Define triangle 'abc' with vertices A, B, C. GeoGebra command: abc = Polygon[A, B, C].
2. Let n = 3.
3. Construct a slider for n. Let n move in increments of 2 (it takes values 3, 5, 7, ...).
4. Let k = (n-1)/(2n).
5. Define: D = B + k*(C-B), E = C + k*(A-C), F = A + k*(B-A).
6. Define: ad = Segment[A, D], be = Segment[B, E], cf = Segment[C, F].
7. Define: P = Intersect[ad, be], Q = Intersect[be, cf], R = Intersect[cf, ad].
8. Define triangle 'pqr' with vertices P, Q, R. GeoGebra command: pqr = Polygon[P, Q, R].
9. Compute the ratio of areas, abc/pqr. GeoGebra command: ratio = abc/pqr.
10. Study dependence of 'ratio' on n. Also, for fixed n, study the dependence of 'ratio' on the choice of the triangle, by dragging the vertices about.

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Some problems from the Olympiads

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

In this edition of "Adventures" we consider some problems from various mathematics contests. The first two are from the very first International Mathematical Olympiad (IMO), held in 1959 in Romania. For those familiar with the IMO in recent years, it may come as a shock to see the vast difference in level between the early years and present times. One longs for the good old days! For those not familiar with the IMO, here is some information about its structure. There are a total of six problems in the Olympiad, administered over two days, three on each day. It is customary for the first and fourth problems to be the easiest in the collection, and for the third and sixth ones to be the most difficult.

In addition we study some problems (numbers 3 and 4 below) from the extensive problem collection available at the following website: http://web.archive.org/web/20040405065644/http://www.kalva.demon.co.uk/index.html. Specifically we use this page: http://web.archive.org/web/20040530211115/http://www.kalva.demon.co.uk/aime/aime83.html.

We state the problems first so you have a chance to try them out on your own.

(1) **Problem 1 of IMO 1959.** *Show that the fraction* $\dfrac{21n+4}{14n+3}$ *is irreducible for every natural number n.*

Note: An "irreducible" fraction is one which is in its simplest form. For example, for $n = 1, 2$ and 3, the given fraction takes the values $\frac{25}{17}$, $\frac{46}{31}$ and $\frac{67}{45}$ respectively. Each of these is in its simplest form.

(2) **Problem 4 of IMO 1959.** *Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.*

(3) *What is the largest prime factor of the central binomial coefficient* $\binom{2000}{1000}$*?* Note: Another notation for $\binom{2000}{1000}$ is ${}^{2000}C_{1000}$.

(4) *How many four-digit numbers with first digit* 2 *have exactly two identical digits?* Note: We refer to numbers like 2001 or 2012.

Discussion and solutions

Problem 1 of IMO 1959. We are asked to show that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n. As noted, the term 'irreducible' means that the fraction cannot be simplified further and is already in its simplest form; no cancellation of factors can be done between numerator and denominator. For example the fractions $\frac{5}{3}$ and $\frac{15}{8}$ are irreducible, but not $\frac{10}{15}$ which can be 'reduced' (simplified) to $\frac{2}{3}$. Hence another way of stating the problem is: *Show that the numbers* $21n + 14$ *and* $14n + 3$ *have no common factors for every natural number* n. Here is yet another way of stating the problem: *Show that the numbers* $21n + 14$ *and* $14n + 3$ *are coprime for every natural number* n.

The word "coprime" may suggest that we will need to find the prime factors of the two numbers involved ($21n + 14$ and $14n + 3$) and then check that there is no overlap in the two sets of primes. However this is an extremely difficult problem! Indeed, if n is some unspecified number, there is no way whatever of finding the prime factors of either $21n + 14$ or $14n + 3$.

Fortunately there is another approach. It rests on a simple fact: *Pairs of consecutive numbers are coprime*. (For example: 9 and 10 are coprime, though both of them are composite numbers. Similarly, 20 and 21 are coprime, as are 25 and 26.) And this in turn rests on another simple fact: *If d is a divisor of two integers a and b, then d is a divisor of* $a - b$. Indeed, if $a = md$ and $b = nd$, where m and n are integers, then $a - b = (m - n)d$. Hence if the consecutive integers n and $n + 1$ share a common divisor d, then d must be a divisor of $(n + 1) - n$, i.e., d must be a divisor of 1, which forces d to be *equal* to 1. It follows that n and $n + 1$ can share no factor other than 1, i.e., they are coprime.

A moment's thought shows that this result can be extended: *If integers a and b have multiples ma and nb which differ by* 1, *then a and b are coprime*. For, if the integer d is a common divisor of a and b, then d is a common divisor of ma and nb, hence d is a divisor of $ma - nb$, i.e., d is a divisor of 1, hence $d = 1$. So a and b are coprime.

So our task reduces to find multiples of $21n + 4$ and $14n + 3$ which differ by 1. But the relevant multipliers are easily found, using the fact that $21 : 14 = 3 : 2$. We get:

$$2(21n + 4) - 3(14n + 3) = 8 - 9 = -1.$$

So $2(21n + 4)$ and $3(14n + 3)$ are consecutive integers, and it follows that $21n + 4$ and $14n + 3$ are coprime, as required.

Problem 4 of IMO 1959. We are asked to construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle. Here, "construct" means: work out a procedure using ruler-and-compass which will accomplish the stated end.

The way we shall solve this is to find out the ratio of the legs using algebra, then to draw a triangle which has the right 'shape' (i.e., it is similar to the desired triangle), and finally to construct the desired triangle.

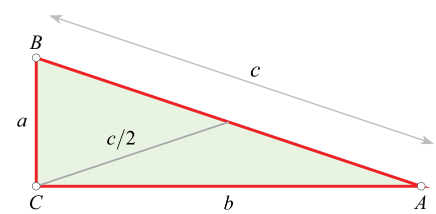


Figure 1.

Let the triangle be as depicted in Figure 1, with legs a and b, and hypotenuse c. Then the median to the hypotenuse has length $c/2$. (Do you see why? Remember that this is a right-angled triangle, so its circumcentre coincides with the midpoint of the hypotenuse.) The condition stated in the problem yields: $ab = (c/2)^2$, i.e., $c^2 = 4ab$. We also have: $a^2 + b^2 = c^2$. The two conditions yield: $a^2 - 4ab + b^2 = 0$. Treating this as a quadratic equation in b we get:

$$b = \frac{4a \pm \sqrt{16a^2 - 4a^2}}{2} = (2 \pm \sqrt{3})a.$$

Hence $b/a = 2 + \sqrt{3}$ or $b/a = 2 - \sqrt{3}$. Let us consider the first possibility. Draw a segment $B'C'$ with length 1, as shown in Figure 2, and a line ℓ perpendicular to $B'C'$ at C'. Draw a ray at B' making an angle of 60° to ray $B'C'$ and let it meet ℓ at D'. Then $C'D'$ has length $\sqrt{3}$. Locate point A' further along ℓ such that $D'A'$ has length 2. Then

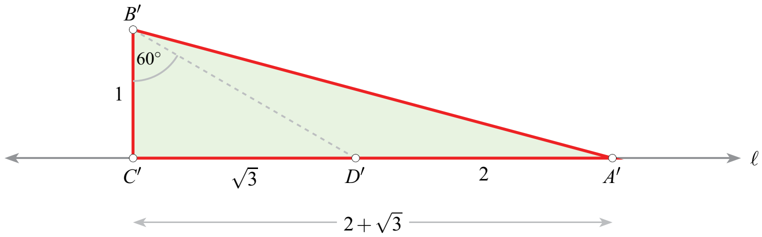


Figure 2.

$C'A'$ has length $2+\sqrt{3}$, hence the right triangle $A'B'C'$ has the right shape.

Having constructed a triangle with the desired shape, we know the angles of the desired triangle. The rest is easy. On a segment AB with the given length c (which will serve as the hypotenuse), we mark off at its two ends the appropriate angles from $\triangle A'B'C'$, using the standard procedure for transferring angles. (We leave the details to you to complete.) This will give us the desired $\triangle ABC$.

You may wonder what happens with the second possibility, $b/a = 2-\sqrt{3}$. We leave it to you to work out the answer, but a strong hint comes from the fact that $2-\sqrt{3}$ is the reciprocal of $2+\sqrt{3}$. So the ratio b/a is the reciprocal of the ratio we got the previous time. What does this tell you about the shape of the corresponding triangle?

Largest prime factor of a central binomial coefficient. We are asked to find the largest prime factor of the binomial coefficient $\binom{2000}{1000}$. By definition we have:

$$\binom{2000}{1000} = \frac{2000!}{1000!\ 1000!} = \frac{2000}{1} \times \frac{1999}{2} \times \frac{1998}{3} \times \cdots \times \frac{1001}{1000}.$$

It should be clear from this expression that every prime number between 1000 and 2000 is a divisor of the binomial coefficient $\binom{2000}{1000}$; for there is nothing in the denominator that can cancel such a prime. So an easy strategy to answer this question is to find the largest prime number between 1000 and 2000. We start from 2000 (i.e., with the largest number in the range) and work our way downwards. We quickly find that 1999 is a prime number. Hence 1999 is the answer to our question.

Counting four-digit numbers. We are asked to count the number of four-digit numbers with first digit 2 having exactly two identical digits (numbers like 2001 or 2012). We subdivide the set of such numbers into two categories: those for which the repeated digit is 2, and for which the repeated digit is different from 2.

If the repeated digit is 2, then the second, third and fourth digits are of the form $2, x, y$ where $\{x, y\}$ is a two element subset of the digit set $\{0, 1, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $\binom{9}{2} = 9 \times 8/2 = 36$. These three digits can be permuted in $3! = 6$ ways. Hence the number of numbers in this category is $36 \times 6 = 216$.

If the repeated digit is different from 2, then the second, third and fourth digits are of the form x, x, y (in some order) where x, y are not equal to 2. We can select x in 9 possible ways, and having selected x, we can select y in 8 possible ways. The digits can be permuted in $3!/2 = 3$ ways. Hence the number of numbers in this category is $9 \times 8 \times 3 = 216$.

Therefore the total number of numbers of the stated type is $216 + 216 = 432$. This is the required answer.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem IV-1-M.1
If the sum of the reciprocals of three non-zero real numbers is zero, is it possible that the sum of the three numbers is zero?

Problem IV-1-M.2
If a and b are integers such that $a + 2b$ and $b + 2a$ are square numbers, show that each of a and b is divisible by 3.

Problem IV-1-M.3
Show that a power of 2 cannot be represented as a sum of two or more consecutive positive integers.

Problem IV-1-M.4
In $\triangle ABC$, one of the mid-segments is longer than one of its medians. Show that $\triangle ABC$ is obtuse-angled. (Note: The mid-segment of a triangle is a segment joining the midpoints of two sides of a triangle.)

Problem IV-1-M.5
Show that in any circle, two non-diametrical chords cannot both bisect each other.

Problem IV-1-M.6
A and B are two boxes. Box A contains 100 white marbles, while box B contains 100 black marbles. We take 10 marbles at random from box A and put them into box B. After this we take out 10 marbles at random from box B and put them in box A. Which is now larger: the number of black marbles in box A, or the number of white marbles in box B?

Problem IV-1-M.7
Let $a_1, a_2, a_3, \dots, a_n$ represent the numbers $1, 2, 3, \dots, n$ subjected to an arbitrary arrangement. Assume that n is odd. Consider the number

$$X = (a_1 - 1)(a_2 - 2)(a_3 - 3) \dots (a_n - n).$$

What can be said about the parity of X. Is this number even or odd?

Solutions of Problems in Issue-III-3 (November 2014)

Solution to problem III-2-M.1 *Amar, Basil, Celia and Dharam are four children. Basil's age is greater than twice Amar's age; the sum of Amar's and Celia's ages is less than Basil's age. Dharam is older than Basil. If Celia is* 6 *years old, and Dharam is* 9 *years old, find Basil's age. (All ages are in whole numbers).*

Let us represent the names by their first letters A, B, C, D. Then $C = 6, D = 9, A + C < B, B < D$. Hence $A + 6 < 9$, i.e., $A < 3$, which yields $A = 1$ or 2. Also, $B < 9$.

Next, $A + C < B$ also gives $B - A > C$ which means $B - A > 6$. Now:

- If $A = 2$, then we get $8 < B < 9$ which is not possible.
- If $A = 1$, then we get $7 < B < 9$ giving $B = 8$. So the required answer is 8 years.

Solution to problem III-2-M.2 *Mary's teacher notes the test scores of* 32 *students in her class. She finds that the median score is* 80 *and the range of the scores is* 40*. The teacher then tells the class that their average score is* 58*. Mary contends that her teacher has gone wrong somewhere. Who is right, Mary or her teacher?* [*Fryer Contest, 2003*]

The average of the class (as claimed by the teacher) is 58. Therefore the total mark is $32 \times 58 = 1856$.

Since the median mark is 80, at least 16 students will have a mark of at least 80; hence the total mark of these 16 students will be at least $16 \times 80 = 1280$.

This implies that the total mark of the remaining 16 students will be at most $1856 - 1280 = 576$; but these 16 students should have actually scored at least $16 \times 40 = 640$. This is so because as the median is 80 and the range is 40, there are students who got at least 80 and so the lowest possible individual score is 40 (we have been told that the difference between the highest and lowest marks is 40).

This contradiction shows that Mary is justified in her suspicion. The teacher has indeed made an error in the computation.

Solution to problem III-2-M.3 *Select* 50 *distinct integers from the first* 100 *natural numbers, such that their sum is* 2900*. What is the least possible number of even integers amongst these?*

To avoid even integers as far as possible, let us try to get the sum 2900 using only the odd integers. The sum of the first 50 odd integers $1, 3, 5, 7, \ldots, 99$ is $50^2 = 2500$. So we need 400 more to get the sum 2900. For this, we have to use only the even integers, keeping intact the total number of integers as 50.

To the extent possible, we try to exchange the smallest odd integers for the largest even integers, in pairs, since 400 is even. So:

- We replace $\{1, 3\}$ with $\{100, 98\}$. This makes the sum $2500 - 1 - 3 + 100 + 98 = 2694$.
- As this is not sufficient we replace $\{5, 7\}$ with $\{96, 94\}$. Now the sum becomes $2694 - 5 - 7 + 96 + 94 = 2872$.
- We are still short by 28, so we try one more exchange. We replace $\{9, 11\}$ with $\{20, 28\}$, and this works: $2872 - 9 - 11 + 20 + 28 = 2900$.

We have used only 6 even integers, namely: $100, 98, 96, 94, 28, 20$. This is the minimal number of even integers needed.

Solution to problem III-2-M.4 *Find the digits A and B if the product $2AA \times 3B5$ is a multiple of* 12*.*

The given product is a multiple of 12, which means it is divisible by 3 and 4. The factor of 4 must come from the term $2AA$ since $3B5$ is odd. Invoking the test for divisibility by 4, we see that $A = 0$, 4 or 8.

- If $A = 0$, then $2AA$ is not divisible by 3, so $3B5$ must be divisible by 3, hence $B = 1, 4$ or 7.
- If $A = 4$, then $2AA$ is not divisible by 3, so $3B5$ must be divisible by 3, hence $B = 1, 4$ or 7.
- If $A = 8$, then $2AA$ is divisible by 3, so the B in $3B5$ can be any digit.

So if $A = 0$ or 4 then $B \in \{1, 4, 7\}$, and if $A = 8$ then B can be any digit. So there are $3 + 3 + 10 = 16$ possible combinations which satisfy the conditions of the problem.

Solution to problem III-2-M.5 *One altitude of a triangle is tangent to its circumcircle. Prove that some angle of the triangle has measure larger than* 90° *but less than* 135°*.*

Let the triangle be ABC. Suppose that AD, the altitude to side BC through A, is tangent to the circumcircle at A. As the tangent lies outside the circle, it intersects BC at a point D on its extension. Assume that D lies beyond B on ray $\overrightarrow{CB}$. Hence $\measuredangle ABC > 90^\circ$. Consider $\triangle ADB$; it is right-angled at D. From the intersecting chords theorem we know that $DB \times DC = DA^2$. (See below for the theorem statement.) We also have $DB < DC$. Hence $DB < DA$. Hence $\measuredangle ABD > \measuredangle BAD$, and so $\measuredangle ABD > 45^\circ$. It follows that $\measuredangle ABC < 135^\circ$. Therefore $\measuredangle ABC$ lies between 90° and 135°. (See Figure 1.)

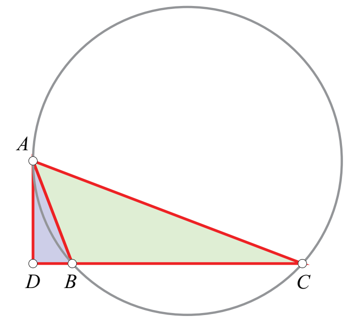
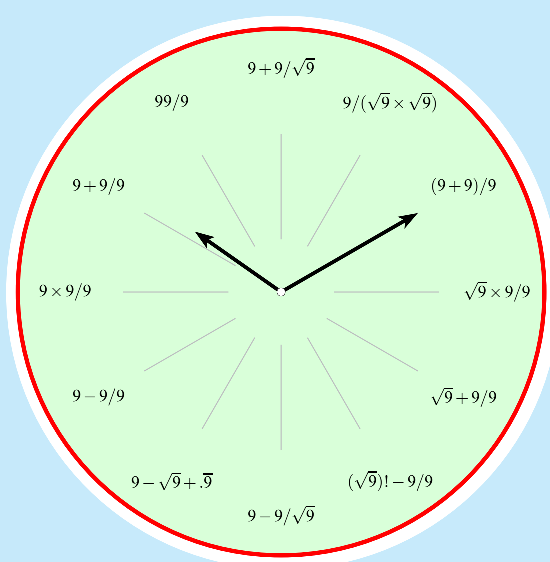


Figure 1.

Note: The **intersecting chords theorem** states the following. Through a point P let two lines l and m be drawn intersecting a given circle Γ at pairs of points $\{A, B\}$ and $\{C, D\}$ respectively. Then $PA \times PB = PC \times PD$. The point P could lie either inside or outside the circle, or on it. If P lies outside the circle and one of the lines, say m, is tangent to the circle at C, then the statement yields: $PA \times PB = PC^2$.

The Versatile 9

It seems that these three 9's have learned the art of cooperatively working together!



Problems for the Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

Problems for Solution

The problems in this set are adapted from the Romanian Mathematical Competitions, 2014.

Problem IV-1-S.1
Let $A = \{1, 3, 3^2, 3^3, \ldots, 3^{2014}\}$. A *partition* of A is a union of non-empty disjoint subsets of A.

(a) Prove that there is no partition of A such that the product of all the elements in each subset is a square.
(b) Does there exist a partition of A such that the sum of elements in each subset is a square?

Problem IV-1-S.2
Let ABC be a triangle in which $\measuredangle A = 135^\circ$. The perpendicular to line AB at A intersects side BC at D, and the bisector of $\measuredangle B$ intersects side AC at E. Find the measure of $\measuredangle BED$ (see Figure 1).

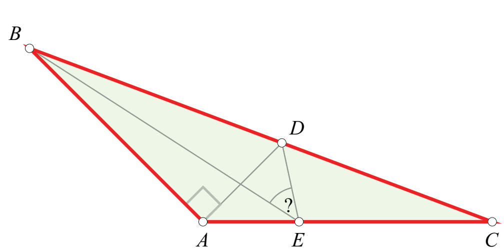


Figure 1.

Problem IV-1-S.3
Determine all pairs (n, p) of positive integers such that

$$\left(n^2+1\right)\left(p^2+1\right)+45 = 2(2n+1)(3p+1).$$

Problem IV-1-S.4
Determine all irrational numbers x such that both $x^2 + x$ and $x^3 + 2x^2$ are integers.

Problem IV-1-S.5
Find all pairs (p, q) of prime numbers, with $p \le q$, such that

$$p(2q+1)+q(2p+1) = 2\left(p^2+q^2\right).$$

Solutions of Problems in Issue-III-3 (November 2014)

Solution to problem III-3-S.1 *If* $(x-y+z)^2 = x^2 - y^2 + z^2$, *prove: either* $x = y$ *or* $z = y$.

First observe that $x^2 - y^2 + z^2 = (x+z)^2 - y^2 - 2zx = (x-y+z)(x+y+z) - 2zx$. Hence if $(x-y+z)^2 = x^2 - y^2 + z^2$, then

$$\begin{aligned} 2zx &= (x-y+z)(x+y+z) - (x-y+z)^2 \\ &= (x-y+z)(2y). \end{aligned}$$

This yields:

$$y^2 - (z+x)y + zx = 0, \qquad \therefore (y-x)(y-z) = 0.$$

Hence $x = y$ or $z = y$.

Solution to problem III-3-S.2 *Prove that the numbers* 10017, 100117, 1001117, ... *are all divisible by* 53.

Let a_n be the n-th number in the given sequence. Then $a_1 = 53 \times 189$. Also: $a_n = 10a_{n-1} - 53$ for each n. Hence if a_{n-1} is a multiple of 53, so is a_n. Since a_1 is a multiple of 53, it follows by the principle of induction that a_n is a multiple of 53 for every n.

Solution to problem III-3-S.3 *Let* $ABCD$ *be a parallelogram. Let the bisector of* $\measuredangle ABD$ *meet* CD *produced at* X *and let the bisector of* $\measuredangle CBD$ *meet* AD *produced at* Y. *Prove that the bisector of* $\measuredangle ABC$ *is perpendicular to* XY.

Let $\measuredangle ABD = 2x$ and $\measuredangle CBD = 2y$ (see Figure 2). Then $\measuredangle BXD = x$ and $\measuredangle BYD = y$. Thus in triangle BDX, $BD = DX$ and in triangle BDY, $BD = DY$. Thus $DX = DY$; and since $\measuredangle XDY = \measuredangle ADC = 2(x+y)$, it follows that $\measuredangle DXY = \measuredangle DYX = 90^\circ - (x+y)$. Hence $\measuredangle BXY = \measuredangle BXD + \measuredangle DXY = 90^\circ - y$.

If the bisector of $\measuredangle ABC$ meets XY at Z then in triangle XBZ, $\measuredangle XBZ + \measuredangle BXZ = (x+y-x) + 90^\circ - y = 90^\circ$. Hence $\measuredangle BZX = 90^\circ$.

Solution to problem III-3-S.4 *Prove that if* $a_1 \le a_2 \le a_3 \le \ldots \le a_{10}$, *then*

$$\frac{a_1 + \ldots + a_6}{6} \le \frac{a_1 + \ldots + a_{10}}{10}.$$

Observe that

$$\begin{aligned} 6(a_1 + \cdots + a_{10}) - 10(a_1 + \cdots + a_6) &= \\ 6(a_7 + \cdots + a_{10}) - 4(a_1 + \cdots + a_6). \end{aligned}$$

Now $6(a_7 + \cdots + a_{10}) \ge 24a_7$ and $4(a_1 + \cdots + a_6) \le 24a_6$. Hence:

$$6(a_1+\cdots+a_{10}) - 10(a_1+\cdots+a_6) \ge 24(a_7 - a_6) \ge 0.$$

The result follows.

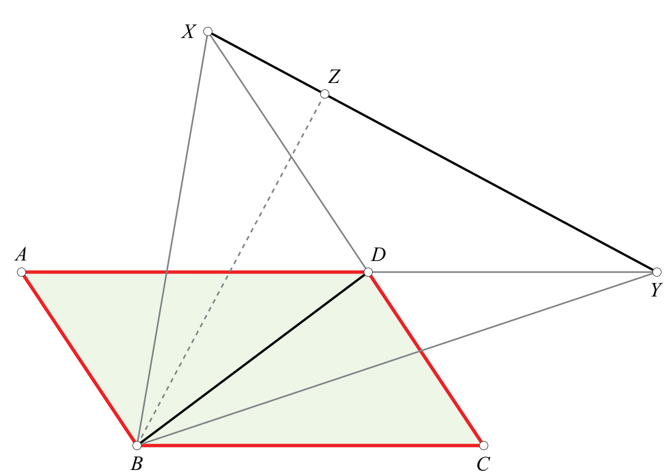


Figure 2.

problem corner

Triangle in a Rectangle

C⊗MαC

Problem. *In rectangle $ABCD$ is inscribed triangle PBQ with P on side AD and Q on side CD. The areas of $\triangle PAB$, $\triangle QBC$ and $\triangle DPQ$ are p, q and r, respectively. Find the area s of $\triangle PBQ$ in terms of p, q, r.* (See Figure 1.)

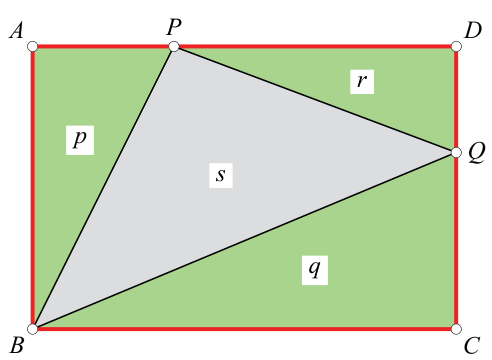


Figure 1.

Try to solve the problem on your own before looking up the solution!

Solution. Assign symbols for lengths as follows:

- $AB = a$
- $BC = b$
- $AP = x$
- $CQ = y$

We redraw Figure 1 with these symbols shown. It is remarkable that merely by defining these symbols and setting out the basic area relations, the answer can be found. Indeed, the "solution derives itself". This is surely one of those problems where the pen or pencil seems to possess its own intelligence.

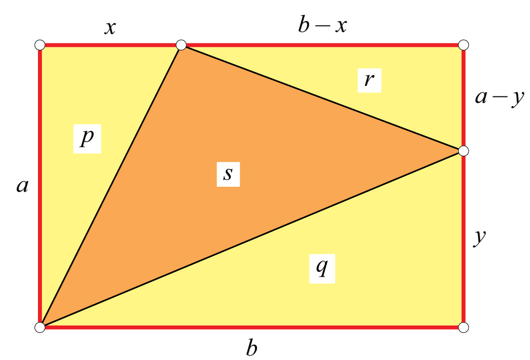


Figure 2.

We have the following relations:

$$ax = 2p,$$
$$by = 2q$$
$$(b - x)(a - y) = 2r.$$

From these we must find the value of ab. The third relation when expanded yields:

$$ab - ax - by + xy = 2r,$$
$$\therefore\ ab + xy = 2(p + q + r).$$

But we have:

$$abxy = 4pq, \quad \therefore\ xy = \frac{4pq}{ab}.$$

Hence:

$$ab + \frac{4pq}{ab} = 2(p+q+r).$$

This is clearly a quadratic equation in the unknown ab. Writing $z = ab$ we have:

$$z + \frac{4pq}{z} = 2(p+q+r),$$

$$\therefore\ z^2 - 2(p+q+r)z + 4pq = 0.$$

Solving this we get:

$$z = (p+q+r) \pm \sqrt{(p+q+r)^2 - 4pq}.$$

The sign ambiguity must be resolved. But z surely must exceed $p+q+r$. Hence:

$$z = (p+q+r) + \sqrt{(p+q+r)^2 - 4pq}.$$

The area s of $\triangle PBQ$ is therefore given by:

$$s = \sqrt{(p+q+r)^2 - 4pq}.$$

This relation may be expressed in more aesthetically pleasing ways, e.g.:

$$(p+q+r)^2 - s^2 = 4pq.$$

As might have been expected, the expression for s is symmetric in p and q. (Why?)

Yet another proof for the "Sum of an Arithmetic Progression" formula

by: Faiz Imam

This proof is based on an article by Martin Griffiths; see [1]. We wish to find the sum S_n of the first n terms of an arithmetic progression (AP),

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a + [n-1]d),$$

where a and d are respectively the first term and common difference of the AP. If $d = 0$ then clearly $S_n = na$.

Now assume that $d \neq 0$. Let $c = a/d$, so $a = cd$. Therefore:

$$\begin{aligned} S_n &= cd + (cd+d) + (cd+2d) + (cd+3d) + \cdots + (cd + [n-1]d) \\ &= dc + [(c+1) + (c+2) + (c+3) + \cdots + (c + (n-1))]d. \qquad (1) \end{aligned}$$

Now we treat c as an integer. Using the formula for the sum of the first k positive integers, we evaluate the sum in line (1):

$$\frac{(c+n-1)(c+n)}{2} - \frac{(c-1)c}{2} = \frac{n(2c+n-1)}{2} \qquad \text{(on simplification).}$$

Hence:

$$\begin{aligned} S_n &= d \times \frac{n(2c+n-1)}{2} = \frac{n}{2} \times \left[2cd + (n-1)d\right] \\ &= \frac{n}{2} \times \left[2a + (n-1)d\right]. \end{aligned}$$

We have got the correct formula for the sum of an AP though we treated c as an integer which it clearly may not be. How is this possible?

REFERENCES

[1] Martin Griffiths, Sums of powers of the terms in any finite Arithmetic Progression, The Mathematical Gazette, Vol. 86, No. 506 (Jul., 2002), pp. 269-271

review

An invitation to math

Edward Frenkel's "Love & Math: the Heart of Hidden Reality" - a review

Mark Kleiner

The eminent mathematician Edward Frenkel has written his autobiography in order to convey to the general audience both the human and professional aspects of mathematics.

He explores the human aspect of mathematics by describing his work on several projects. He shows that, in addition to perseverance and hard work, solving a difficult mathematical problem requires imagination and original ideas, and that beauty and elegance usually characterize important mathematical results. The creative process of a mathematician in many ways resembles that of an artist or musician and generates the whole gamut of emotions, the most important of which is love. For it is a genuine love for mathematics that carries a mathematician through the often frustrating and emotionally demanding work. The book describes the erotic film "Rites of Love and Math," which was inspired by the short film "Rite of Love and

Keywords: *Mathematics, beauty, elegance, Langlands program, Shimura-Taniyama-Weil conjecture, Rosetta stone*

Death" based on a story by the great Japanese writer Yukio Mishima who both directed the film and starred in it. Frenkel invented the plot and played the Mathematician in "Rites of Love and Math." The Mathematician creates a formula for love, but realizes that the formula can be used for both good and evil. To prevent it from falling into wrong hands, he hides the formula by tattooing it on the body of the woman he loves. The idea is that "a mathematical formula can be beautiful like a poem, a painting, or a piece of music" (page 232). The rite of death plays an important role in the Japanese culture. The title of Frenkel's film suggests that mathematics plays an important role in the world culture. He describes his motivation to create the film as follows. "In popular films, mathematicians are usually portrayed as weirdos and social misfits on the verge of mental illness, reinforcing the stereotype of mathematics as a boring and cold subject, far removed from reality. Who could want such a life for themselves, doing work that supposedly had nothing to do with anything?" (page 229). And here is his answer to his own question. "In truth, the process of creating new mathematics is a passionate pursuit, a deeply personal experience, just like creating art and music. It requires love and dedication, a struggle with the unknown and with oneself, which elicits strong emotions. And the formulas you discover do get under your skin, just like the tattooing in the film," (page 233).

Frenkel sheds light on the professional aspect of mathematics by introducing the reader to the Langlands Program, which he perceives as "a Grand Unified Theory of Mathematics because it uncovers and brings into focus mysterious patterns shared by different areas of mathematics and thus points to deep, unexpected connections between them," (page 70). Robert Langlands, currently Professor Emeritus in the School of Mathematics at the Institute for Advanced Study in Princeton, initiated the program in the late 1960s as a conjecture that hard problems of number theory can be solved by using methods of harmonic analysis. Since then, the scope of the program has significantly grown due to contributions of other mathematicians and of physicists, and at this time the Langlands Program is a program of study of analogies and interconnections between four areas of science: number theory, curves over finite fields, geometry of Riemann surfaces, and quantum physics. The first three on the above list are different areas of mathematics. The modern formulation of the part of the Langlands Program concerning these areas was arrived at by the end of the 20th century. The realization that the mathematics of the Langlands Program is intimately connected to quantum physics via mirror symmetry and electromagnetic duality came in the 21st century, around 2006-2007. Frenkel personally participated in achieving the latter breakthrough. His book presents a first-hand account of the work done.

Why is the Langlands Program important? Because it brings together several areas of science that cover a wide range of research, are sufficiently far apart, and use different methods and techniques. Each of the areas has been developing in its own way motivated by results, questions, and conjectures deemed important within it. Once an area is related to another one, the experts in each of these two areas face questions and ideas, as well as methods and techniques, transplanted from the other area, which are often new and unexpected. The situation is similar to a transfusion of fresh blood to a person whose energy level has been stagnant for a while. The interconnections are beneficial to each of the areas involved.

The book discusses in detail precursors of the Langlands Program.

One of them is the Shimura-Taniyama-Weil conjecture, which played an important role in the proof of Fermat's Last Theorem by Andrew Wiles and Richard Taylor in 1994. The theorem, a statement in number theory, had been the most famous open problem since 1637, when the French mathematician Pierre Fermat wrote about it on the margin of an old book he was reading. The reasons for the fame were, on one hand, the simplicity of the statement: there are no positive integers $n > 2$ and x, y, z satisfying

$$x^n + y^n = z^n,$$

and, on the other hand, the number of years the problem had resisted the efforts of people to solve it. Familiarity with middle school mathematics is more than sufficient to understand the problem, and many generations of people, both experts and amateurs, had confused the simplicity of the statement with the ease of proof. The first main step towards the proof was made in 1986 by Ken Ribet who showed that Fermat's Last Theorem follows from the Shimura-Taniyama-Weil conjecture. The second main step was the proof of the latter by Wiles and Taylor in sufficient generality. The Shimura-Taniyama-Weil conjecture, which establishes a connection between elliptic curves over the field of rational numbers and modular forms, relates number theory to harmonic analysis. It is a special case of the Langlands Program.

Another precursor of the Langlands Program is the Rosetta stone proposed by the French mathematician Andre Weil in 1940. The name Rosetta stone is a reference to the famous ancient stele, on which essentially the same text was written in three different languages and which was used to decipher ancient Egyptian hieroglyphs. Weil's Rosetta stone establishes an analogy between number theory, the theory of algebraic curves over finite fields, and Riemann surfaces. He visualized these three areas of mathematics as three columns of Table 1 (below).

Weil's main idea was that the middle column is a bridge between the left and right columns, i.e., the results and arguments from the left column can be translated to the right column and vice versa via the middle column. Based on his Rosetta stone, Weil formulated three conjectures, the proof of which "greatly stimulated the development of mathematics in the second half of the twentieth century," (page 104). Weil's Rosetta stone is another special case of the Langlands Program.

Frenkel formulates "four qualities of mathematical theories . . . : universality, objectivity, endurance, and relevance to the physical world," (page 228), of which objectivity seems to be the only controversial one in that it claims that mathematical concepts and ideas inhabit the Platonic world of mathematics that exists independently of the physical reality or the mental world. He quotes several famous mathematicians and physicists who support his point of view.

The book is aimed at the reader who may have no background or interest in mathematics or even hates math. Using plain language and examples from everyday life and minimizing the use of formulas, Frenkel presents highly nontrivial mathematical facts. At the same time, if the reader gets interested and decides to delve into the described theories more deeply, the Notes contain many formal definitions, rigorous proofs, and heuristic arguments. The author also describes numerous applications of mathematics to support his point that an advanced society should be mathematically literate.

The book discusses mathematics in the context of the author's mathematical career and, therefore, places the reader in the Soviet Union between years 1983, when Frenkel was in the ninth grade of high school, and 1989, when at the age of 21 he became one of the first four recipients of the Harvard Prize Fellowship after his mathematical papers had been smuggled abroad and *perestroika*, which was initiated by General Secretary of the Communist Party Mikhail Gorbachev, had lifted the iron curtain. Frenkel has lived in the United States since 1989.

Although the focus of Frenkel's narrative is his learning and doing mathematics, as well as his interaction with other mathematicians, by default he paints a picture of life in the Soviet Union.

Number Theory	Curves over Finite Fields	Riemann Surfaces

Table 1

The reader learns about the widespread anti-Semitism when Frenkel describes how he, the son of a Russian mother, was denied admission to *Mekh-Mat* of *MGU (Moskovskiy Gosudarstvennyy Universitet)*, that is, the Department of Mechanics and Mathematics at the Moscow State University, because he had a Jewish father. He discusses careers of other Jewish mathematicians, some of them truly exceptional, who were mistreated by the Soviet system. The reader also learns about prominent Soviet mathematicians, both Gentile and Jewish, who recognized Frenkel's talent, mentored him, and helped him find his place in mathematics. They did this of their own volition and without any remuneration.

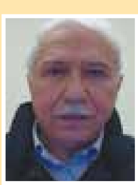
I cannot resist adding a personal touch here. I graduated from *Mekh-Mat* of Kiev State University in 1969, being one of the less than ten graduates with straight A's. Out of the graduating class of 175, more than half were admitted to the graduate school at Kiev State, but the door was shut for me because of my Jewish descent. I got a job as a computer programmer, continued doing mathematics in my spare time, and defended my Ph D thesis in 1972. An academic position was beyond my reach, so I immigrated to the United States in 1979. That was possible due to the Jackson-Vanik amendment to the US federal law, which forced the Soviet government to lessen the restrictions on the emigration of Jews.

The reader gets a glimpse of the great mathematician Israel Gelfand and his weekly seminar at MGU, "an important mathematical and social event, which had been running for more than fifty years and was renowned all over the world," (page 61). Many outstanding mathematicians considered it an honor to meet Gelfand and give a talk in the seminar. It was open to both Gentiles and Jews. Since most of the latter were not formally affiliated with MGU, the seminar provided "a safe haven" where they could develop as mathematicians. The core of Gelfand's seminar comprised the "Gelfand mathematical school," of which Frenkel was a member.

The book also mentions correspondence schools, organized by famous Soviet mathematicians, Gelfand among them, that reached out to talented high school students who lived outside of major cities and did not have access to special mathematical schools. In contrast, very few prominent American mathematicians get directly involved with teaching high school students.

To summarize, the book is a lively account of the life and work of a prominent mathematician.

The issues raised are diverse enough to make the book interesting to a wide range of readers, from a person having little or no knowledge of mathematics to an experienced professional mathematician.



MARK KLEINER was born in the Soviet Union in 1946. He received his MS and PhD from Kiev State University. He worked later as a computer programmer in Kiev, but in his spare time he worked on representations of algebras. He was a postdoctoral student at Brandeis University (USA), and has been a professor of mathematics at Syracuse University (USA) since 1981. He may be contacted at mkleiner@syr.edu.

Error Regretted

Dear Readers,
In the previous issue of At Right Angles, November 2014, we had missed acknowledging the contributor for the article "Folding a 45º, 60º, 75º Triangle from a square sheet of paper in 6 easy steps" on page 24.

The article was contributed by ***Roopika Jayaram****, (Sahyadri School (KFI), Chas Khaman Dam, Tiwai Hill, Rajgurunagar, Pune – 410513. E-mail: roopikajayaram@gmail.com)*

The error is regretted.

The Closing Bracket . . .

Two themes stand out in this issue: that of proofs without words, or PWWs as they are generally referred to, and that of algorithms for school-level arithmetical operations and manipulations and their hidden basis. The former serves to bring out the 'fun' and visually appealing side of the subject by designing compact and pretty pictures that drive home some point. The latter points to the deep interconnectedness of the subject by showing that familiar operations and algorithms of elementary arithmetic typically have roots that lie deep within the structure of the subject.

Very few students get to see PWWs, and perhaps fewer still are challenged to construct or devise pictures of their own which serve to illustrate some notion. Yet it appears to me that it may be a wonderful way to learn something of value by getting them to ponder the matter at hand, slowly and with a sense of leisure, searching for ways of depicting patterns and structures. It is necessary to bring in the element of slowness into math education (and into all of education, more generally), and devising pictures is one possible way to approach this challenge.

Similarly, very few students get to see and experience the interconnectedness of the subject. High quality exposition may be needed to bring out this factor, but one possible approach is through non-routine problem solving. The problems have to be chosen with care, but they can serve as excellent platforms for bringing in multiple strands together and seeing interconnections.

Here is a quote from Paul Halmos, the great Hungarian-born US mathematician who was an equally great teacher: *It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts*. PWWs would surely qualify for the description "much more than facts"; so also the matter of unearthing interconnectedness. Of course that phrase includes a lot more than that: it also includes the experience of conjecturing, of experimentation, of testing your conjectures, of devising counterexamples, of possibly finding that your conjectures are wrong, of proving them. It is only when one experiences the subject in this complete sense that one begins to see the wholeness of the subject. All students of mathematics have the right to experience the subject in that way, and we surely have to make it so for them. (Here's another quote from Halmos: "Mathematics is not a deductive science — that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork." Paul Halmos certainly had a way with words.)

Halmos was a great proponent of the so-called "Moore method" for teaching mathematics. But let's leave that theme for a future issue.

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
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13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

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At Right Angles
A RESOURCE FOR SCHOOL MATHEMATICS

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Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

TEACHING GEOMETRY-II AT THE PRIMARY LEVEL

PADMAPRIYA SHIRALI

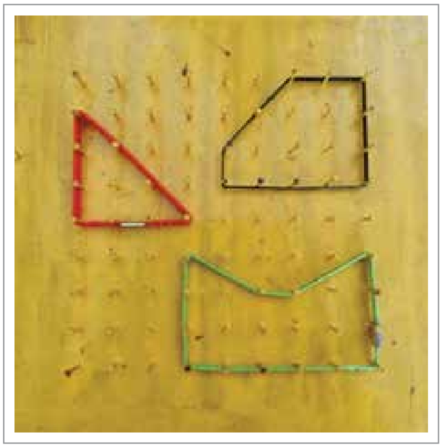
At Right Angles
A Resource for School Mathematics

[...Continued from the previous issue]

In many ways, the teaching of geometry when approached in the right way holds an immense potential for learning the art of seeing and observing. As one begins to play around either with a plain square paper by folding it in different ways or connecting dots in a dot paper one begins to see a variety of shapes emerge. Personally I have always found it to be a pleasurable and enriching experience as every student that I meet sees and notices shapes in his or her own special way. There seems to be a lot of scope for developing an eye for seeing, for improving observational skills and for bringing out one's creativity. Also amongst all the topics in mathematics it is in the teaching of geometry that a teacher can use a hands on approach to a high degree to discover properties of shapes, the manner in which they behave when they are moved around, rotated, slid, the way they can be dissected and reassembled to form new shapes, etc. In fact, once the teacher introduces an activity within no time children will develop their own variations of it and begin to make their own explorations. There is no greater reward that a teacher can obtain than that.

CLASS TWO

As children move into higher grades one may continue to use the same materials used in the earlier years (shapes, geo-boards, straws, dot paper, paper plates, tangram sets) but one raises the level of challenge in each activity.



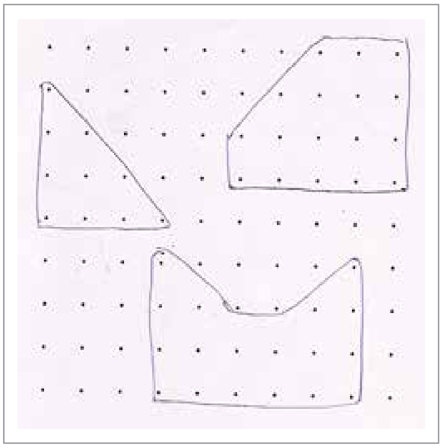


Figure 1

Geo-board

For instance: in activities involving usage of a geo-board, the teacher can require the children to record the shape they have created on a geo-board on a dot paper. Simple as it seems, many children find it a complex task. They need to visualise the corresponding points of their shape on the geo-board on the dot paper. They need to have a sense of the size of the shape and its orientation; the number of pegs enclosed vertically and the number of pegs enclosed horizontally; the straightness of the sides, etc. This leads up to the idea of mapping which comes much later.

PATTERN CREATIONS ON DOT PAPER

Teacher can now follow up the geoboard activities with more dot paper activities.

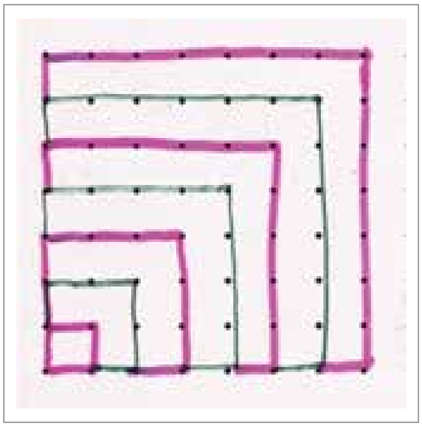


Figure 2

They can use square dot paper to create shape patterns. They can draw series of same sized squares in different colours, alternating series of squares and rectangles or alternating series of squares and triangles, series of squares of increasing size (concentric or starting from the same corner), and so on (Figure 2).

They can try more complex patterns involving squares bordered with triangles on all sides.

They can make patterns which result in a circular or a closed formation, and they can also make patterns which continue to expand.

In the process of building these patterns they use their intuitive understanding of symmetry, similarity, tessellations etc.

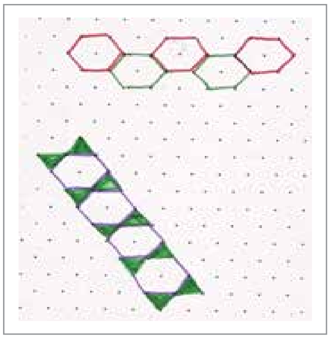


Figure 3

In a similar manner, triangular dot paper can be used to create many patterns using triangles and hexagons (Figure 3).

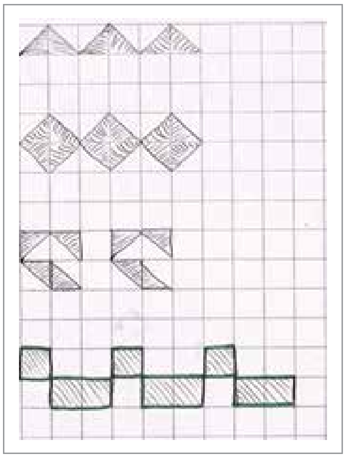


Figure 4

The teacher can give some patterns of shaded squares (Figure 4) and ask the children to copy them. This too requires them to observe small areas carefully and builds spatial abilities.

PAPER FOLDING AND SYMMETRY

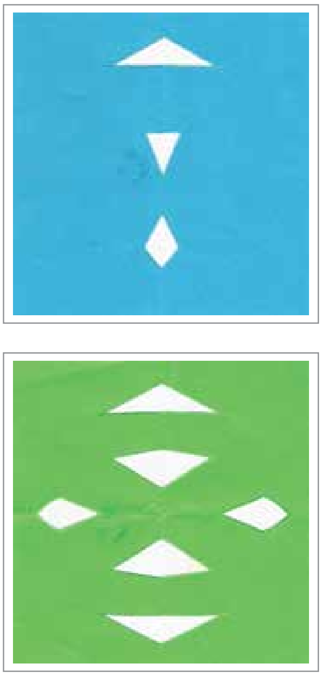


Figure 5

Let them use paper folding and cutting to create symmetrical shapes (Figure 5).

Initially they can explore cutting papers made with one fold and see the shapes that emerge. Later they can make two folds, vertical and horizontal, and observe the shapes thus created.

Through experimentation let them discover the different types of cuts and the shapes that result due to these cuts.

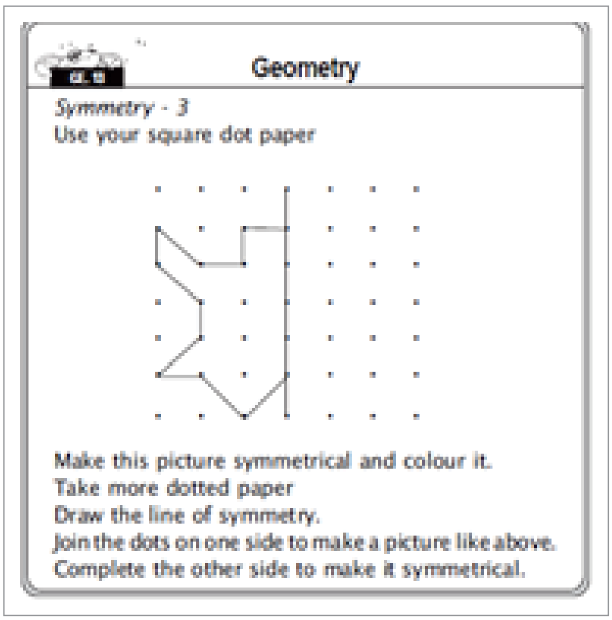


Figure 6

Continue to explore symmetry by making half-complete designs. Let children copy them onto dot paper and complete the other side (Figure 6).

Various simple rangoli patterns can be taught and created.

They can do further work on symmetry through ink designs or paint designs on folded paper.

TANGRAMS

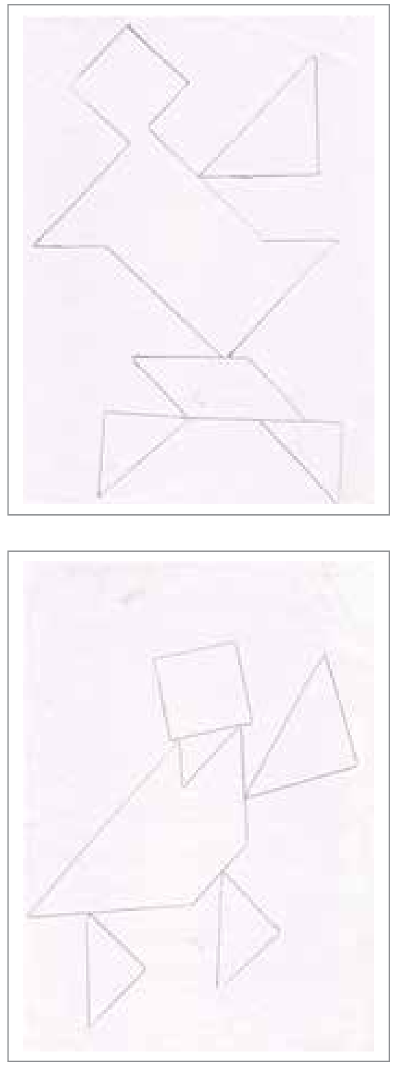


Figure 7

Show them some designs (select simple ones as shown in Figure 7) and require them to build these designs. Initially give designs with the outlines drawn to size on a large chart paper so that children can fit the tangram pieces onto the outline. Children often solve them using intuition. In the process of deducing the pieces to be used to build the given design, they will need to observe the design carefully: the corners, slant of the lines, size of the shape, etc.

CLASS THREE

3-D STRUCTURES

How similar? How different?

Discuss the 3-D structures they see around in the class room. Some shapes are regular 3-D shapes: cuboid, sphere, etc. But some shapes we use resemble cuboids or cylinders closely but are not exactly cuboids or cylinders. *Example:* A water bottle is often not a complete cylinder but narrows down at the opening. Discuss: "In what way is it different from a cylinder?" A tiffin box may look almost like a cuboid but may not be exactly a cuboid. Similarly, a compass box is generally curved at the corners. We need to recognise that while doing this activity children are closely observing, comparing and contrasting both the similarities and the differences.

Building skeletal models of 3-D shapes

Normal sturdy drinking straws or paper straws can be used to build skeletal models (Figure 9). Children can be taught to roll small paper sheets tight and make them into paper straws. They make excellent material for various activities. Children can build cubes, cuboids, prisms, pyramids etc. with them by joining them with small rubber tube connector pieces.

Ask: "How are these different from solid cubes or cuboids?" Introduce simple vocabulary like faces, edges, corners, straight lines, slant lines etc. (See Figure 10.)

Ask: "Where do these two faces meet?" "Where do these three faces meet?" "Can you show me two faces which do not meet?" "Which face is opposite to this face?" "Which solid shape has most edges?" "Which solid shape has edges of the same length?" "Which solid shape has edges of different lengths?" "Which solid shape has only square faces?" "Which solid shape has rectangular faces?"

Ask questions about where two edges meet. For example in the case of a tetrahedron ask: "How many V joints do you see?" "How many L joints do you see?" "Do you see any other kind of joints?"

Let them contrast shapes like cuboids and cubes with shapes like prisms and pyramids (naming is unnecessary).

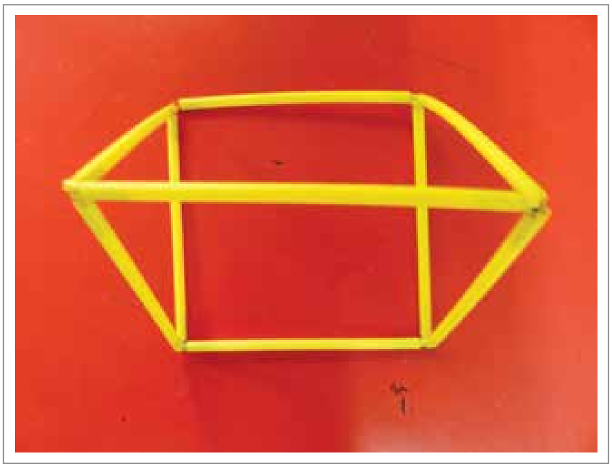


Figure 9

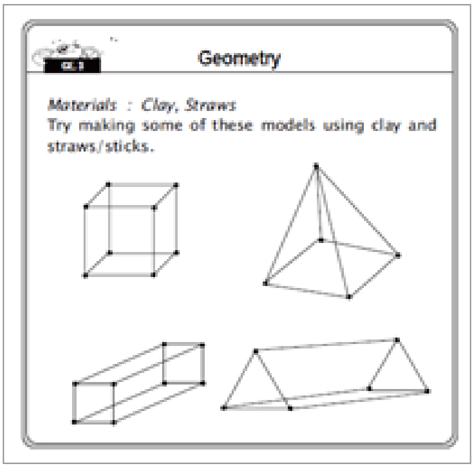


Figure 10

Building 3-D shapes from 2-D shapes

Let them stack up many wooden triangle pieces and see the solid shapes they can form (e.g., a prism).

Let them stack up many wooden rectangle pieces and see what shapes they can form (e.g., a cuboid).

Let them stack up many square wooden pieces and see the solid shapes they can form (e.g., a cube or a cuboid).

Give children old glossy magazine paper. Ask: "Can you turn this into a cylinder shape?" If they roll up the paper and stick the edges together, it turns into a hollow cylinder. Let them make hollow cylinders of different diameters.

PAPER FOLDING

Let them feel their way into the activity. The more they work with paper, the greater will be their skill and accuracy in paper folding. Ask: "In how many ways can you fold a square paper into half?" They may fold a square vertically or horizontally or diagonally into half and make the edges match. Ask them to describe the shape that has formed after the paper is folded. Let them explore other ways of folding a square paper. "What shape do you get when you make a fold in one corner?" "How does the shape change when you fold one more corner?" "What happens when you fold all four corners?"

Now pose the same question about a rectangle. They may try to do the same with a rectangle and find that if they fold a rectangle from one corner to the opposite corner the edges do not get aligned, unlike a square. Here too they can experiment with folding different corners to see the shapes that emerge.

Similarly they may see that a triangle can only be folded in some ways to get a half triangle. They may also notice that different types of triangles fold differently. It is important that all these are discovered by children through play and occasional suggestions from the teacher (when the children seem to have run out of ideas).

Let children try to make regular shapes by creasing rectangular pieces of paper.

Tangrams

Designs can now be given in scaled down size. Children may be able to build the shapes by looking at the designs. The teacher can also give designs which use only a few of the pieces.

The teacher can also ask questions which require children to use the tangram pieces to create other regular shapes. Ask: "Take two small triangles. Can you make a square with them? Can you make a bigger triangle? Can you make any other shape?"

Figure 11

POSITIONS AND MAPS

Positional instructions: Introduce relevant vocabulary like left, right, turn, etc. Give children a sequence of instructions which they need to follow, say to go from one place to another place in the school. Use simple instructions: left, right, straight, turn around, etc.

Let them also describe a sequence of their movements using correct positional language.

Simple maps with concrete objects: Let children create a simple map of their class room by using rectangles, squares and circles to represent the teachers' table, their benches, perhaps a shelf or a dustbin. The teacher can discuss positional relationships, what is to the left or to the right, what is close to the teacher's table, what is further away etc.

This can be followed by simple route related problems. They can explore ways of getting from point S ('start') to point F ('finish') in a 3 by 3 grid as shown in Figure 12.

How many ways can you get from S to F if only vertical/horizontal moves are allowed? Which route is the longest? Which one is the shortest?

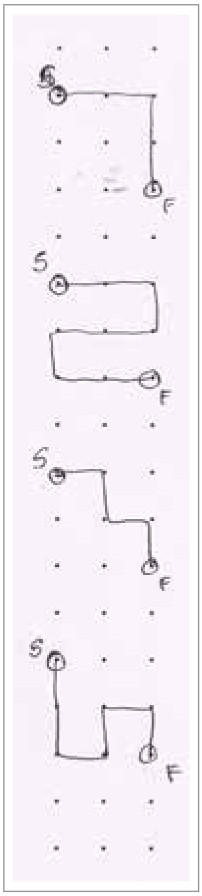


Figure 12

Game: Visualising shapes

The first student traces a regular shape like a square, rectangle or circle with his finger on the back of the second student who then feels the tracing, follows the turns, counts the sides and so on to visualise and name the shape.

CLASS FOUR

At this point one can plan activities to get them ready to start formal geometry in the fifth year. The concept of angle and measurement of angle is one chief area where children face problems. The teacher should do plenty of activities involving measurements of corners of varied objects in the class using a square paper as a measure and many activities involving turn (using arms of the body, or arms of the clock).

3-D shapes

Let them study various 3-D objects (cubes, cuboids, cylinders, cones, triangular prism, spheres) by identifying the number of faces, shapes of the faces, the number of edges and the number of corners of each solid shape, and recording them in a table form.

NETS

Let children use nets of cubes and cuboids to build them. The teacher can draw the nets on the board and children can draw them using a square shape or a rectangular shape for making the net (Figure 13). At this stage they will not be able to make an accurate drawing using a scale.

Drawing 3-D shapes on triangular dot paper

Show them how to draw a cube on dot paper (Figure 14). They can also try drawing cuboid, sequence of connected cubes (towers) etc.

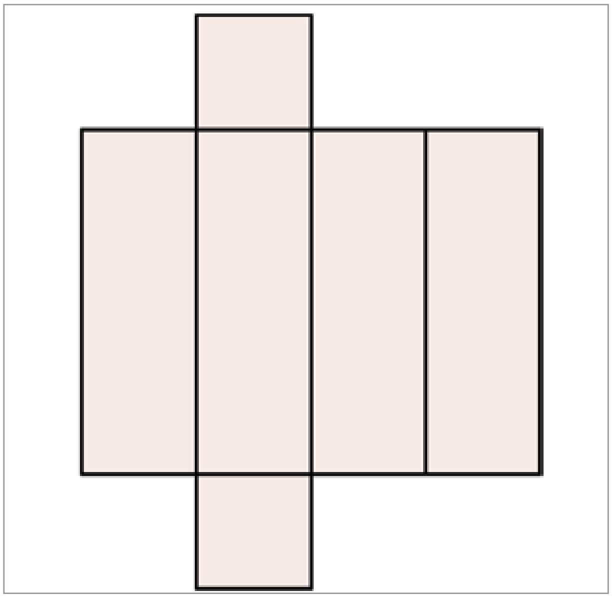


Figure 13

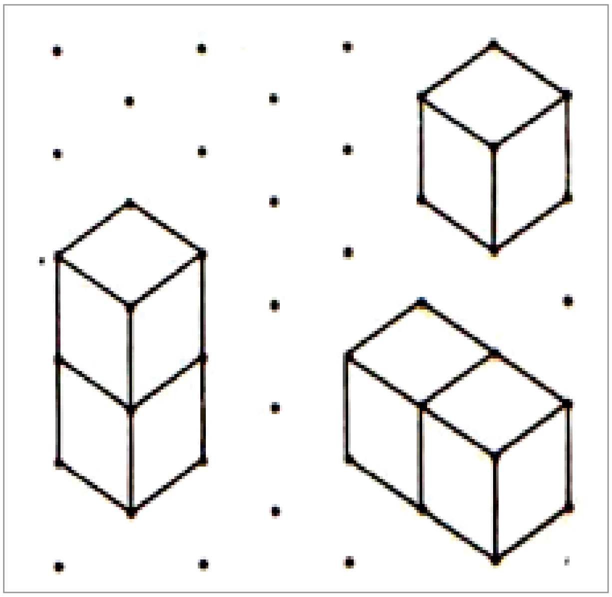


Figure 14

GEO-BOARD ACTIVITIES

Let them build composite shapes which fit together and describe them in terms of corners and lengths of the sides etc.

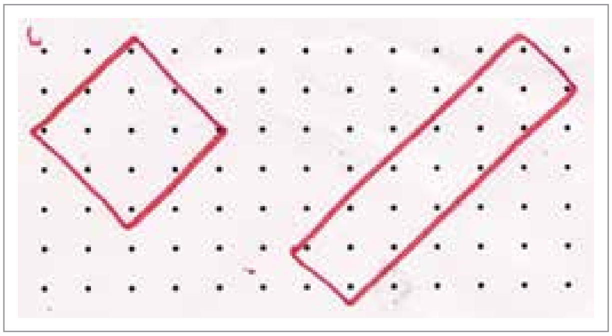


Figure 15

Now require them to build regular shapes linking pegs diagonally as well. Ask them to create a square or a rectangle (Figure 15). To get the sides of a rectangle or a square straight and parallel involves greater challenge now. It needs finer observation skills.

Follow up geoboard activities with dot paper based explorations.

Suggested explorations

Use a 3 by 3 dot grid and connect the dots with straight lines (horizontal or vertical or slant) to make different shapes (see Figure 16).

How many different shapes can you make? Name some of them.

Which shape has the maximum number of edges?

What is the minimum number of dots needed to make a shape? What shape was this? What is the maximum number of dots that you used? What shape did you get?

Could you make squares of different sizes? Could you make rectangles of different sizes? How many types? Could you make triangles of various sizes? How many types?

2-D shapes

Materials: Hexagons, large squares, small squares, different types of triangles, circles, pentagons and octagons.

Figure 17

Let children use some shapes to create symmetric designs and closed patterns as shown in the picture (Figure 17).

"What shapes can you make with two triangles and with three triangles?" Draw as many of them as you can on a sheet of paper (draw the outlines of the composite shapes formed).

"Can you make a bigger triangle using the small triangles?" (The bigger triangle need not be of the same shape as the smaller ones.)

"Can you make a triangle using square shapes? Why or why not? Can you make a square shape using triangles? Why or why not?"

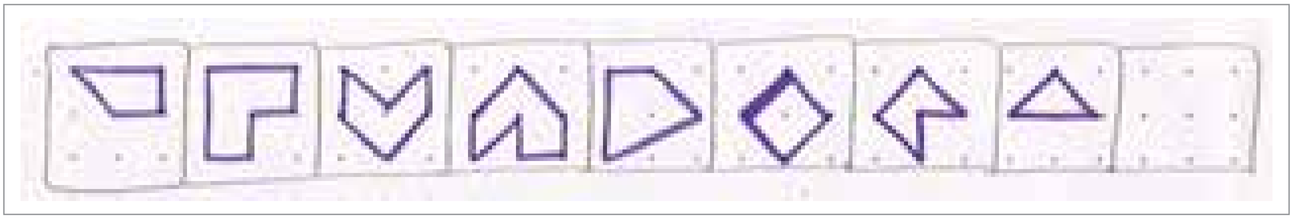


Figure 16

Outlining activities with shapes

Children can trace the outline of a shape on plain paper and check the number of ways the shape can fit into its own outline.

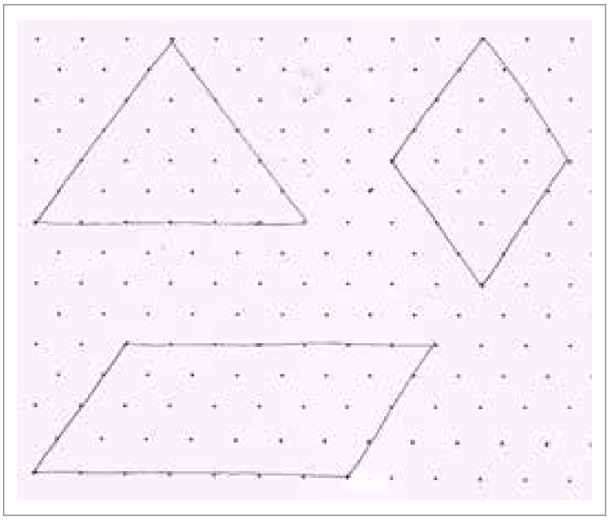


Figure 18

Spotting shapes in other shapes: Draw some shapes as shown in Figure 18.

Ask: "Can you see a hexagon and some triangles in these shapes?"

Tessellations (tiling patterns)

Children can be shown tessellating patterns found on paths and walls, or pictures of tessellations. They can play around with various shapes to create patterns which tessellate. Let them make a table of shapes that tessellate and shapes that do not. Through experimentation they can find shapes that fit together and shapes that do not fit together. They can also make tessellations on dot paper.

Ask: "Which shapes cover a surface without leaving any gap?" "Which shapes leave a gap?" "What is the shape of that gap?"

PAPER FOLDING

Start with a square sheet of paper. Ask the children to show you that the given shape is actually a square. They will demonstrate by folding one edge onto the opposite edge (horizontal fold) and one edge onto the adjacent edge (diagonal fold). They can make a quarter fold to demonstrate that all the angles are equal.

Let them explore folding a paper (square or rectangular) in different ways. Some possible ways can be to fold a paper at the centre horizontally but not at the centre while folding vertically. What shapes are formed now? Let them make a border fold on all four sides of a paper. Open and see the shapes that have formed. Let them try different slant folds which can result in a pointed roof shape or a boat shape.

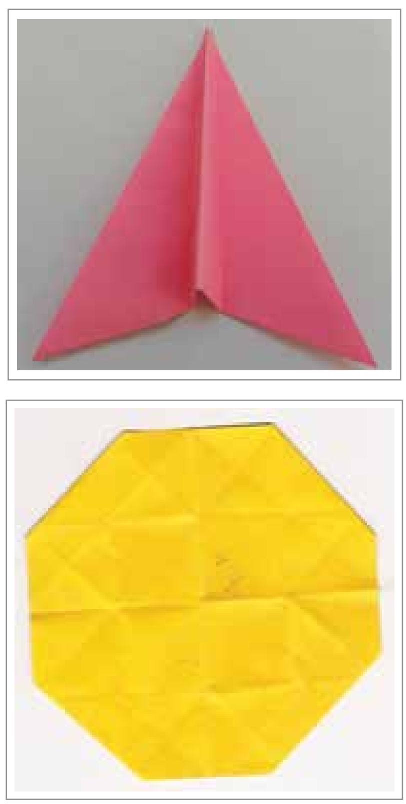


Figure 19

One can also pose a question the reverse way. Show them a sheet you have already folded (not involving more than two or three steps) and ask them to figure out the folds needed to create that shape.

Creating different shapes using square paper: Show children a parallelogram, a rhombus, a hexagon and a pentagon and ask them if they can make these shapes out of a square sheet of paper. They can experiment with folding the corners to create these shapes.

Folding a circular sheet of paper (e.g., paper plates) generates interesting shapes and designs.

They can try to create a square, a rectangle and a triangle. Also show the children a rhombus and a hexagonal shape and ask them if they can fold the sheet to get those shapes.

Angle as a corner

Let children use a square piece of paper to check the corners of different shapes in the class (teacher's table, floor tiles, text books, triangle pieces, other polygon shapes).

They can make a list of the shapes in the class, those which are square corners, those which are less-than-square corners and those which are more-than-square corners. Teacher may decide to use formal vocabulary: right angle, acute angle and obtuse angle.

Game: "What is my shape?"

Materials: Set of different shapes in pairs (rectangles, squares, triangles of different shapes, parallelograms, rhombi, trapeziums, pentagons, hexagons, circles).

Teacher separates out the two sets, keeps one set and gives the other set to the children. One of the children picks up any one shape from the teacher's set.

The child now describes the shape in terms of its corners alone to the other children. (Ex: My shape has one square corner and two corners less than a square.)

The child may also give information about sides and corners. (Ex: I have 4 square corners and 4 sides, but I am not a square.) Or she may describe the slant nature of the edges. (Ex: I have 4 sides and two of my opposite sides are slanted.)

The other children must now deduce what shape it could be and thus pick up its twin from the other set.

This activity trains the children to look for special properties of the shapes. They soon realise that if they do not describe a special property there can be many shapes with that description.

I have found this game to be useful in sharpening children's observation and logical skills.

Angle as a turn

Connect two sticks with a rubber tube piece so that it forms a rigid bendable L shape.

Let them fix one stick firmly and turn the other around noting the angle between the sticks as they make a quarter turn, half turn, three-quarter turn and full turn.

This activity can also be used for comparing two angles. They can place it over one angle by aligning the arms of the L shape with the arms of the first angle. They can then place it over the second angle and find which angle is the bigger one.

Students can be given a worksheet with many pairs of equal angles drawn in a jumbled up manner. They can then identify pairs of equal angles using the above approach.

Angle hunt

Children can be asked to look for acute angles in their surroundings. Let them examine chairs, clocks, hinges, fan blades, etc.

Lines

Through paper folding activities one can discuss lines, points.

Ask children to fold a paper in such a way that many lines pass through one point. The point can coincide with a corner of the square, it can lie on one side of the square, or it can lie in the interior of the square.

The concept of parallel lines and perpendicular lines can be brought in.

Lines on dot paper

On a sheet of dot paper, circle two random points which are at a distance from one another. Ask: "If these two points are connected with a straight line, which other points will lie on the line?" Circle three random points. Ask: "Will all these lie on one line?"

Line drawings

Line drawing is a very interesting activity. It helps in developing the skill of using a scale, aligning it and marking points exactly. It requires them to identify pairs of corresponding points. It can be done initially on dot paper and later on plain paper.

Steps for construction

Draw a pair of straight lines of length 10 cm at right angles to each other, as shown in Figure 20. The point where they meet is numbered 0. The vertical line is numbered from top to bottom, 10 to 0. The horizontal line is numbered from left to right, 0 to 10. Now join 10 on the vertical line to 1 on the horizontal line, 9 on the vertical line to 2 on the horizontal line, 8 to 3, 7 to 4, and so on. This activity produces a pleasing curved line effect.

Once children learn this technique, they generally come up with various ways of drawing the initial lines (V shape, triangular shape, 'plus sign' shape, etc) to create beautiful designs.

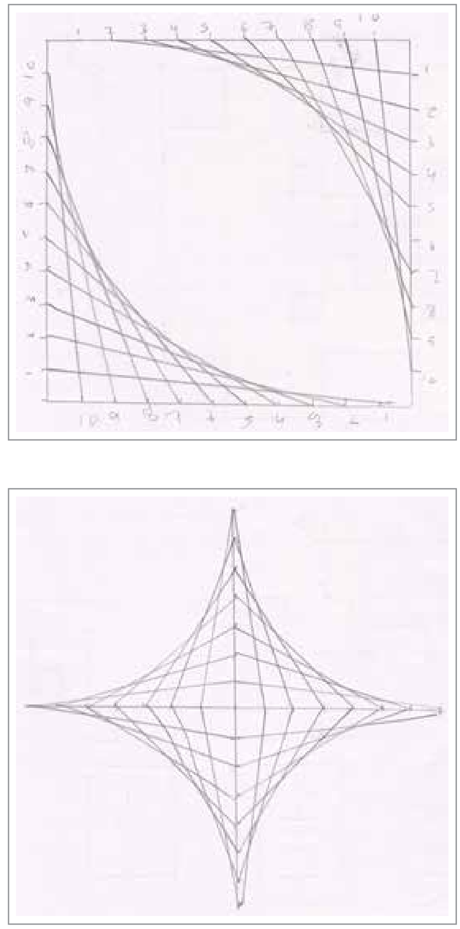


Figure 20

Shape dissections

Cut a large cardboard square into 3 to 5 pieces like a jigsaw puzzle. Let children put them back together to form a square.

This can also be tried with a rectangle, triangle or a circle.

Challenge

Draw a rectangle on a card board. Draw the diagonals and cut it along the diagonals to create 2 pairs of identical triangles. These 4 triangles can be rearranged in different ways to generate outlines of shapes. Let the children figure out the way the four triangles fit on the outline.

Map

Use maps of the neighbourhood of the school for children to map the route from home to school (provided it is not too far). Let them describe the route using positional language.

Point to a place in the map and ask: “If you are standing by this building, is the bank to your left?” “What building lies at the first right?” “Do you turn left or right at this point?” “How will you describe the route to the bus stand?”

Blindfold Game

Another fun activity to build positional sense in children is to get children to do a blindfold walk in a room. One child gives instructions to the other who is blindfolded in the form 'Take two steps forward, turn right, take three steps, now turn left' etc. The purpose of the game is to reach from a start point to an end point, avoiding some obstacles in the room.

Many children's magazines bring out attractive puzzles like mazes, matchstick puzzles, route problems, spotting hidden cubes in a stack of cubes, etc. All these assist in building spatial and logical skills.

Figure 20

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com