The Teaching and Learning of Mathematics

Rohit Dhankar

The Introduction

In this paper I shall assume that the most important concern for a curriculum of any subject is to chart out a route of learning which is most conducive to the development of understanding in that particular subject area. Many factors have to be considered in search of such a route. Difficulties encountered by the learners should be one such factor. This becomes specially important when a need for reconsideration of curricular framework is felt, as the difficulties faced by learners become one of the important criticism of the existing curricular framework. Another important factor is the nature of the subject area. These two aspects and their interplay are taken as the main themes of this paper.

One can hardly deny that there is a lot of frustration among learners of mathematics at the primary and middle school level. It is not so apparent in the departments of mathematics in colleges and universities because most of the students who go there feel that they can handle the jinn. But in schools mathematics is a compulsory subject and a majority of students dislike it, mainly because they find it to be difficult to understand. Mathematics is the subject which brings down the pass percentage in the broad examinations (at least in the case of the Board of Secondary Education, Rajasthan), mathematics has the biggest share of private tutors and a lot of school-tears are shed on the account of this subject.

Before we jump to suggest remedies in terms of improved curriculum and teaching methodology, perhaps it would be a good idea to ask: why is it so? Why mathematics gets more than its share of frustration in the classrooms?

Particular manifestation of a general problem

I would claim that there is a general lack of understanding among the students running across the curriculum. There may be many reasons for this, including inadequate attention paid to concept formation and learners experience. It so happens that this lack of understanding when reaches a critical level becomes an insurmountable difficulty in mathematics and effectively blocks any semblance of further learning, while in other subject areas pretence of further learning can be maintained.

To check the veracity of this statement one can ask an average graduate in physics to explain the difference between “mass” and “weight”; or, alternatively, one can ask a graduate in geography what kind of climate shall we have in Delhi if the
axis of rotation of earth were vertical to the plane of the Earth’s orbit around the sun? If one is not prepared to be satisfied with strings of words however well formed and impressive they might be, but insists on the clarity of concepts and relationships between them the concepts would be found to be hazy and relationships more or less nonexistent. This would imply that even these subjects are learnt only by rote and not properly understood.

It might seem that I am diluting the issue by pointing to these problems. But my purpose here is exactly the opposite. I am trying to point to the iceberg the tip of which is seen as ‘frustration-in-mathematics-teaching/learning.’

But the question remains: why this lack of understanding is much more prominently manifests itself and generates greater difficulty in learning mathematics? In order to answer this question we have to take note of a few relevant features of understanding and the nature of mathematics.

The understanding and its forms

What I am embarking upon right now may seen to be a digression. But I would like to claim that it is not a digression at all, rather is the most direct way to attack the problem. To my mind the specific activities and teaching methodology acquire their meaning and significance only in a properly understood theoretical background. And unless that background is made clear the specific techniques always remain in a danger of becoming rituals. While on the other hand some one who is clear about the theoretical context and aims is likely to find her own specific techniques. And when one is thinking about the curriculum these considerations are an absolute must. Thus I believe Richard Skimp is right when he says, “until we have a better understanding of understanding itself, we shall be in a poorer position either to understand mathematics ourselves, or to help other people to do so.”

The term understanding is used here to mean the ways of interpreting, organizing and analysing experience. This is done by the means of concepts, conceptual structure and organizing principles. The concepts could be handled only through symbols. These concepts as well as the symbols denoting them have to be “publicly rooted” to borrow P.H. Hirst’s phrase. Also the interpretation and analysis have to be judged for their rightness, truth or adequacy. The criteria for such justification, again, have to be publicly shared. Therefore we can say that understanding is the system of publicly shared concepts, conceptual structures, organizing principles and validation procedures employed to interpret, organize and analyse experience. As many thinkers have pointed out, this system taken as a whole is not a monolithic structure.
On examination the ways of understanding would turn out to be distinguishable from each other and could be classified in broad categories. These broad categories of ways of understanding could be termed as “forms of understanding”. There could be more than one classifications of knowledge depending on the chosen criteria. The following discussion borrows the basic idea from P.H. Hirst’s classification.

What is termed here as ‘forms of understanding’, Hirst has called “forms of knowledge”. For “developed forms of knowledge” Hirst lists four “related distinguishing features” which could be abridged and paraphrased as follows:

(i) The concepts, which are peculiar to a form.
(ii) Relationships between the concepts and the logical structure generated by these relationships, which is distinctive to a form.
(iii) The kind of expressions generated within a form of understanding.
(iv) The truth criteria and justification procedure are also distinctive to a form.

The first three are actually overlapping. The logical structure and expressions can be considered as implied by the nature of the concepts central to that form. In connection with mathematics the systems of notations are very important. Therefore, for the present purpose I will organize the discussion under the following headings:

(i) The concepts,
(ii) The conceptual structures and the notations, and
(iii) The validation procedure.

The forms of understanding are distinguishable ways of interpreting and organizing experience; they should not be confused with the ‘subjects’ in a school curriculum. Hirst gives a list of eight different forms of understanding, which could be further subdivided. A particular subject may be a subdivision of a form of understanding (like physics is a subdivision of science which is a basic form of understanding) or it may be a field of study composed of portions borrowed from different forms of understanding (like geography which has borrowed portions from science, mathematics, history etc.). Also the fundamental forms of understanding themselves overlap and borrow from each other. Like science borrows a lot from mathematics, still one can more or less accurately point out the portions in science that basically belong to mathematics.

However, mathematics happens to be one of the basic forms of understanding and a particularly distinguishable at that. Therefore, the above-mentioned
distinguishing features could be used more profitably to characterize the nature of mathematics.

The distinguishing features of mathematics and their implications

In this section I will try to investigate how mathematics differs from other forms of understanding. The emphasis shall be on the features related to teaching/learning of mathematics.

1. The Concepts

All concepts are abstract entities. They have no physical properties like shape, size, colour, sound, test etc. This makes them rather elusive objects difficult to handle. To overcome this difficulty we attach them to sound and visual symbols. The sound symbols attached to the concepts can be called their names. Here we should make a clear distinction between a concept, its symbol and the objects, which are examples of that concept. The word ‘bottle’ is a symbol of the concept, which is evoked in our minds when we hear this word. The word is not the concept, the image, which this word evokes, is. The particular bottle we may have in front of us is not the concept either. It is just an instance of the general idea, which encompasses all the bottles, which we have seen in the past and are likely to see in the future, that ‘general idea’ is the concept. The visual symbol of a concept consists of the marks we make on paper etc. to evoke that concept in the mind of a reader.

Our first concepts are formed on the basis of the sensory experience of the outside world. “To have a concept”, as Dearden tells us, “is to be in possession of a principle of unity according to which a number of things may all be reported as being the same or as being of one kind.” First we discover these principles of unity in concrete things and their experiences. And thus form concepts like bottle, glass, pot, bucket, sweet, saur, bitter, etc.; these could be called “primary concepts”. But we do not stop here and see a principle of unity when concepts like bottles, glass, pot, buckets etc. are put together and form a more general concept, that of containers. Or on the basis of sweet, sour, bitter, etc.; we can form the concept of taste. The principle of unity according to which bottle, glass etc. could be seen as of one kind can be grasped only when one already has these concepts. Thus it is abstracted from other concepts, such concepts could be called “secondary concepts”.

The secondary concepts are farther removed from the sensory experiences and therefore are more abstract. Concepts could be very abstract and very far removed from experience. It was said that concepts are peculiar to a form of understanding. Now we can try to understand what is peculiar to mathematical concepts.

The report of the committee on MLL published by NCERT mentions some concepts under the heading of “Readiness of Primary Mathematics” before listing the
concepts and competencies for class I. Which means the said document considers these concepts to be the least abstract concepts in mathematics. These concepts are: size, length, thickness, weight, volume, shape, colour, position, quantity; and relationships are: ‘smaller than’, ‘bigger than’, ‘the same as’, ‘heavier’, ‘heaviest’, ‘near’, ‘far’ and ‘nearest’.

All these are every day concepts. None of them is actually peculiar to mathematics. The report rightly considers them pre-requisites to the learning of other mathematical concepts. And we note that all of them are secondary concepts. This is not meant to be a criticism of the report; actually they are right, as I said above, in listing these concepts in the beginning. The point I am trying to make is that before we can even start learning mathematical concepts we have already come to a point where direct experience alone is not sufficient to form even the prerequisite concepts. Thus, though it is no proof, we can conclude that almost all the concepts in mathematics are higher order concepts.

Secondly, the mathematical concepts form rather strictly defined hierarchies. A missing link in the hierarchy is sure to hamper formation of all concepts above it in that hierarchy. Which means there are no short cuts in learning of mathematical concepts.

Thirdly, the mathematical concepts have no real instances in the outside world. They do not refer to any physical entity, force or phenomenon. Some of them, like a square, seem to refer to certain things or shapes on paper. But the mathematical square is an ideal confirming strictly to clearly laid down criteria and the shape marked on the paper is more a symbol of the mathematical square rather than the real thing. As a result, they are very precisely defined. Mathematics does not like fence sitters and ambiguity mongerates.

This has certain implication for the curriculum as well as the teaching methodology.

C1 Since the mathematical entities are no where to be encountered in the world and they are so abstract “mathematics can not be learnt directly from the everyday environment but only indirectly from other mathematicians, in conjunction with one’s own reflective intelligence” this implies the curriculum should provide for carefully organized experiences from where mathematical generalizations can start with the help of the teacher.

C2 “The concepts of higher order than those which people already have can not be communicated to them by a definition, but only by arranging for them to encounter a suitable collection of examples”. This means plenty of carefully
selected examples should be used in the text-book and definitions should be used only to consolidate a concept when that is already formed.

C3  In a mathematics curriculum the logical priority of concepts should be clearly worked out and order of learning of concepts, wherever implied by a careful conceptual analysis, should be followed.

C4  Since mathematical concepts are abstract there is a danger of their becoming mere symbols, therefore, in the early stages a lot of relevant experience should be provided for in the curriculum itself. At the same time they should be formed in such a way that they can become free from reliance on the apparatus, then only shall they become genuinely mathematical in nature.

C5  A variety of examples should be chosen so that the concept learnt can become free from the incidental features of the examples.

C6  Precision and clarity should be recognized as a value in mathematics curriculum from the beginning.

2. The Conceptual Structures and the notations

The concepts have connections with other concepts. A class of such connections is implicit in being a part of a hierarchy of which we have talked earlier. Each concept can be a part of several hierarchies at different levels. Apart from these hierarchical relationships concepts have various types of other relationships as well. For example the relationships of equivalence, being a successor, being opposite of, etc.

Complex networks of such relationships connect concepts to form conceptual structures. As the concepts are building blocks of understanding, these conceptual structures are basic tool of learning. When we encounter a new situation we try to interpret it according to our existing conceptual structures. If it fits in one of these structures we have interpreted and connected it with our existing understanding and thus have learnt something new. If it does not fit into any of the existing structure we feel confused and a need to reconstruct some of our conceptual structures or to form a new one has raised.

If we do not have adequate conceptual structures and are unable to reconstruct or form new ones we do not learn. Often inadequate conceptual structures are formed which are not generalizable and therefore cause problems in learning new things. For example a child who has memorized numerals up to hundred, in spoken and as well as written form, without having or connecting the notions of units, tens and place value the memorised pattern, has formed and inadequate conceptual structure. He is almost sure to have difficulty in learning additions with carry over. And when, after a
lot of drilling, he learns to manage the additions with carry over is not likely to be able to generalize the concept of carryover to include the concept of ‘borrowing’ in subtraction. While a child who has learnt to write numerals with an adequate understanding of place value etc. should easily form the idea of carryover and is more likely to be able to reconstruct the same conceptual structure to assimilate the idea of borrowing.

The learning of rules without understanding causes formation of inadequate conceptual structures which are not extendable, and therefore, likely to create learning problems.

One characteristic of an adequate concept and adequate conceptual structure is that they should not be considered sacrosanct and/or God given. When structures are formed with reflection the learner sees them as tools he himself has carved out. Therefore when a need to reconstruct or discard a conceptual structure arrives he can confidently do so in favour of a better one. While when they are handed down to him the learner shall find it difficult to discard them when need be.

Implications for curriculum and teaching methodology:

C7 The rote memorizing of rules should be avoided at all costs. They should have no place in the curriculum.

C8 The learner should be helped to form conceptual structures keeping in mind the long term needs of understanding mathematics. That provides a criteria for choosing curricular content.

C9 Topics should be arranged in such a way that the conceptual structures unfold gradually and effortlessly.

C10 Generally the curricula include material related to utilitarian value of mathematics, like skills and competencies which might be useful in everyday problem solving. Material which has no particular direct utilitarian value, like finding patterns in the number system etc., should also be included in the curriculum. This will give the child an opportunity to develop new conceptual structures and that of discovering hidden beauties of mathematics.

The relationships between mathematical concepts are all logical relationships. They are implicit in the concepts. Depend on nothing outside the system itself. Therefore there are no contingent factors involved. Thus the relationships can be defined as clearly and precisely as the concepts themselves. Since the conceptual structures are dependent only on the concepts and these relationships this quality of precision and clarity is transmitted to them as well. But the conceptual structures are
often very complex. A lot of manipulation of concepts and structures is required and every thing is highly abstract on top of all this. At first sight it seems that the situation is hopeless.

But mathematicians have developed systems of notations which pack a lot of information together, maintain the clarity and precision of the system and allow accurate manipulation of concepts. This they achieve by creating symbols for every thing including the rules of manipulation. And do it so successfully that one can carry on the routine manipulations automatically, so much so that a mathematician can afford to let go of the concepts attached to symbols, perform his operations only on the symbols and still follow all the rules correctly and decipher the meaning contained in a string of symbols at any stage he likes.

Thus a formal language is developed which can express the mathematical structures in remarkably regular patterns. But this achievement of mathematics can also turn against the understanding of mathematics. The whole structure has regular patterns; the symbolized rules are easy to remember and apply; the children are good at memorizing. While on the other hand organizing experiences and undertaking conceptual analysis are difficult tasks; often unpleasant, demanding concentration and discipline, and building of concepts may consume time. Therefore a teacher who teaches mathematics as a system of meaningless symbols that is mechanically manipulated can achieve ‘good’ results in terms of marks obtained and still may spend little time and energy.

And that is how mathematics is being taught in most schools today. This approach works well initially. Then volume of things to be rote learnt increase, becomes too much to handle, the child looses the track of meaning. Further learning stops and the child gets frustrated.

On the other hand, the learner who keeps track of meaning through adequate conceptual structures and practices operations to the point of being automatic can keep his mind free from the routine and pay more concentrated attention to the new things he is learning now. And therefore learns faster.

Implications for curriculum and teaching methodology:

C11 The curriculum should draw the learner’s attention to the notation as well and should aim at developing ability to distinguish a good notation system from a bad one.

C12 The teacher should try that the children can do simple routine operations automatically but should not let it degenerate into mechanical manipulation.
In automatic manipulation the learner can stop and explain what he is doing and why at any stage, while in a mechanical manipulation he can not explain this.

3. **The Validation Procedure**

   The final criterion for the truth of a statement depends on the form of understanding to which the statement under consideration happens to belong. It may be authority of some one who knows better in language usage and literature. It may be empirical data enmeshed into a lot of logic and theory in science. What is the final criterion in mathematics?

   Fortunately it is easier to answer this question for mathematics than for other forms of understanding. In mathematics the final appeal is always made to the internal consistency the system.

   The mathematicians use verifications as well as proofs to demonstrate the consistency of a statement with the accepted results within the system. Verification can be used profitably to establish the truth of a statement which asserts something particular. Like “x=4 is the solution to the equation 4x+5=21”. But for general statements like “the sum of all the angles of a triangle is equal to two right angles” a proof is necessary. A verification in the later case is neither possible nor sufficient. Not possible because no ‘real’ triangle is actually a mathematical triangle and also because one can always demand a higher degree of accuracy than our instruments are capable of. Not sufficient because one can always say “yes, this particular triangle confirms to the rule but the next one which I will draw may not.”

   Therefore, in this case one has to derive the result rigorously from the accepted axioms with the help of accepted rules of inference. We can say that the mathematicians force the septic to accept the conclusion by the sheer power of logic.

   The mathematicians have achieved a very higher degree of agreement based on purely rational grounds. Still there is always a possibility of some one like Mr. Tortoise in Lewis Carroll’s dialogue “What the tortoise said to Achilles” who accepts that:

   (A) Things that are equal to the same are equal to each other.

   (B) The two sides of this Triangle are things that are equal to the same.

   And does not accept that

   (C) The two sides of this Triangle are equal to each other.
Such a person, as Achilles rightly remarks, “had better abandon Euclid and take to football”.

The point I am trying to make is that if some one fails to ‘see’ the logic here no amount of empirical data or authority is going to help such a person. Fortunately for human thought; and especially for mathematics; such a person is an extreme rarity (I haven’t met one so far).

Therefore we can sum-up that: (a) the truth of a mathematical statement can not be established on the basis of authority and empirical data; (b) a mathematical statement has to be proved on the basis of accepted axioms and the accepted rules of inference; and (c) some one who does not have the concepts, conceptual structures, axioms and the rules of inference can not ‘see’ the truth of a mathematical statement.

**Implications for the curriculum and methodology**:

C13 The curriculum should include the ideas of consistency and proof.

C14 As soon as it becomes meaningfully possible the assumptions and rules of inference should be made explicit.

C15 The teacher should not force the child to accept something as correct or wrong if the child does not ‘see’ the logic behind it.

C16 The atmosphere in the class should not be authoritarian.

**The Conclusion**

I have tried to emphasise some of the important aspects of teaching and learning mathematics in this paper. The implications for curriculum are based on the logical grounds alone. Actually the content of curriculum can not be divided on logical grounds alone, many other things have to be considered. But I believe that the logical priorities should be considered important and one should not violate them without sufficiently compelling grounds.

I also believe that the teaching and learning of mathematics can not be improved in isolation. It is not possible that the reason and understanding are encouraged in mathematics and all other subjects are taught in an authoritarian manner.

Every curriculum has epistemological assumptions. Generally these assumptions are not stated explicitly. Therefore they remain unanalysed and one can not even be sure whether the curriculum is internally consistent in terms of its epistemology. Therefore a curricular statement should explicitly state the key epistemological principles it has assumed.
References

4. An alternative criterion could be seen in *Realms of Meaning*, by P.H. Phenix.

(1993?)

Rohit Dhankar
Digantar, Todi Ramjanipura, Jagatpura, Jaipur – 302025
Telephone: 0141 750310, 0141 750230
Email: digantar@datainfosys.net