Features

From Regular Pentagons to the Icosahedron via the Golden Ratio

Part I

SHASHIDHAR JAGADEESHAN

Introduction

This series of articles will explore an amazing connection between three different objects in mathematics: the regular pentagon, the Golden Ratio and the icosahedron. Obviously, if the Golden Ratio is involved then the Fibonacci sequence can't be far behind and, if the icosahedron is, so is its dual the dodecahedron!

In the first of these articles we will begin with the question of how to construct regular polygons, and restrict our attention to the regular pentagon and some of its properties. The regular pentagon serves as a doorway to a veritable treasure house of interconnected mathematical ideas and it never fails to astonish me.

At the outset I would like to acknowledge that almost all the ideas discussed here can be found in [3]. In this series of articles we expand on some of the ideas and take a few digressions along the way.

Like many things in mathematics, construction of regular polygons comes with a rich history. As a part of school geometry we teach students geometrical constructions. While the standard school geometry box has many instruments like the protractor and set square, the Greeks were particularly interested in those constructions that can be performed using only a straight-edge (a ruler with no markings for measurements) and compass. So for example, many constructions we do in the school syllabus like bisecting an angle and drawing the perpendicular bisector of a given line segment are examples of straight-edge and compass constructions. Apart from some of these easy and obvious constructions it's amazing what one can do with these tools. For example, one can construct a square having the same area as that of a lune, i.e., a plane figure bounded by two circular arcs! For an excellent account and proof of this fact,

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I would urge you to read [2]. It is not clear why the Greeks imposed these constraints on themselves. Perhaps it was due to the notion of Plato that the only 'perfect' figures were the straight line and the circle, or it was an intellectual game, with very precise rules. It must be noted here that the Greeks themselves did not always restrict themselves to these tools and did not hesitate to use other instruments.

Although the Greeks were extremely successful in many constructions using only a straight-edge and compass, there were some questions that they were not able to resolve. The most famous of these are called **the three problems of antiquity**:

1. *The duplication of a cube,* or the problem of constructing a cube having twice the volume of a given cube.
2. *The trisection of an angle*, or the problem of dividing a given arbitrary angle into three equal parts.
3. *The squaring of a circle*, or the problem of constructing a square whose area is equal to that of a given circle.

Many professional mathematicians, amateurs and cranks have spent several hours trying to solve them and have not been successful, because these problems are impossible to solve! That is right; no matter how smart you are or how hard you work, no matter if you are a Ramanujan or the latest math God, you cannot solve these problems using the straight-edge and compass constraints. Moreover, we can prove that this is so! The notion of impossibility in mathematics is fascinating but to really understand the proofs requires knowledge of some abstract algebra. For those interested please see [4] and [5].

Constructing regular polygons and the regular pentagon in particular

Apart from the three problems of antiquity, the Greeks were also interested in constructing regular polygons. With their ingenuity they were able to construct polygons with 3 sides, 5 sides and in general, regular polygons with 2^m sides, for $m \geq 2$. They also knew that if they could construct polygons with r sides and s sides, then they could construct one with rs sides, provided r and s are relatively prime (that is, r and s have GCD 1). If we were to list all the regular polygons they could construct with $n < 100$, the list would be regular polygons or n-gons with n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, 40, 48, 60, 64, 80, 96. So they had no clue how to construct several regular n-gons, with even small n = 7, 9, 11, 13, 14, 17, 18, 19!

One of the reasons Gauss is revered so much is that in 1796, at the age of 19, he demonstrated how to construct a regular polygon of 17 sides using straight-edge and compass! This after nearly 2000 years of stagnation as far as construction of regular polygons was concerned. Gauss showed further that a regular polygon of n sides can be constructed whenever n is prime and is of the form $n = 2^k + 1$ for some integer $k \geq 0$. These primes are actually called *Fermat Primes*. You can check that 17 is one such.

According to the historian Archibald [1], this discovery made a great impression on Gauss.

> *"But the extraordinary discoveries of Gauss, while yet in his teens, greatly extended this class of polygons and settled for all time the limits of possibilities for such constructions. In this connection the discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses was not only one of which Gauss was vastly proud throughout his life, but also, according to Sartorius von Waltershausen, one which decided him to dedicate his life to the study of mathematics."*

So from our discussion we see that we can construct a regular polygon of n sides provided $n = 2^{i+2}$ for $i \geq 0$ or $n = 2^i p_1 p_2 \cdots p_j$, where the p_i, are distinct Fermat primes.

Recall, Fermat primes are primes of the form $n = 2^k + 1$ for some integer $k \geq 0$. The question that naturally comes up is, what if n cannot be expressed in the above manner? Well! Pierre Wantzel in 1837, proved that if n is not of this form, then it is **impossible** to construct a regular polygon with n sides.

Let us turn our attention to humbler pie! All of you are probably familiar with the construction of an equilateral triangle, a square and a hexagon. The construction of a regular pentagon however is not so straightforward. This is probably because the interior angle of a regular pentagon is 108°, and it is not so easy to construct this angle.

How would we approach such a construction?

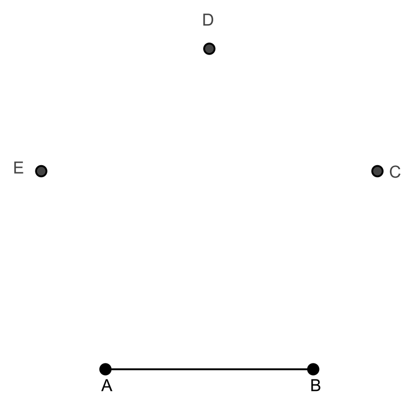

Figure 1

We are given two points A and B and a fixed length $|AB|$ and we need to find points C, D and E, so that $ABCDE$ forms a regular pentagon, that is, all the sides and interior angles are equal. Remember you have only a straight edge and compass. So, we can transfer lengths, draw circles with fixed radii and join two points to form a line. If we think a bit, we will see that if we knew how to construct the diagonal length $|AC|$, we would be done!

We illustrate below:

The point C is the intersection of circle 1 which has centre B and radius $|AB|$ and circle 2 which has centre A and radius the length of the diagonal. We can see now how we can find points D and E using a similar method. Of course we have assumed that the diagonals of a regular pentagon are all equal. We leave this for the reader to prove!

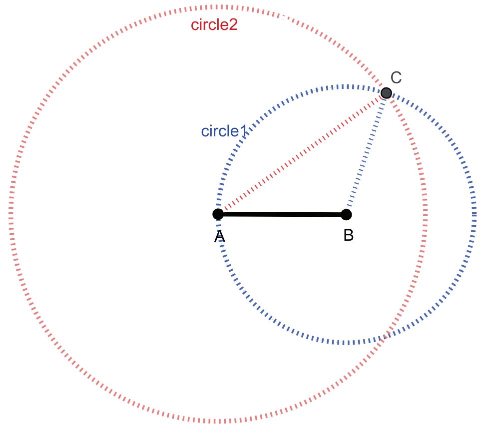

Figure 2

So the problem of constructing a regular pentagon has been reduced to that of constructing the diagonal of a regular pentagon. For the sake of convenience we shall assume that $|AB|$ is of unit length and we shall denote the length of the diagonal by ϕ. So what is ϕ?

The first step is to figure out all the angles and lengths in the regular pentagon shown in Figure 3.

This follows from the fact that the interior angle of a regular pentagon is 108° and that the triangles $\triangle ABC$, $\triangle ABE$ and $\triangle ECD$ are isosceles. If you stare at Figure 3 long enough you will get it!

We now see that $\triangle ABF \sim \triangle ECD$ giving the following ratio

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

and hence the quadratic equation $\phi^2 - \phi - 1 = 0$.

Using the quadratic formula we find

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

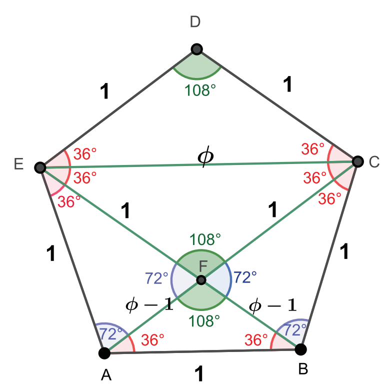

Figure 3

and because $\phi > 0$, we have $\phi = \frac{1+\sqrt{5}}{2}$, the Golden Ratio!

Coming back to the question of constructing a regular pentagon, we said it reduces to that of being able to construct ϕ. We have just shown $\phi = \frac{1+\sqrt{5}}{2}$. Is $\frac{1+\sqrt{5}}{2}$ constructible? In school geometry we learn that if x is constructible then so is $\sqrt{x}$ (if you have not learned this then bug your teacher to show you!), therefore $\sqrt{5}$ is constructible and so is $\frac{1+\sqrt{5}}{2}$ and hence the regular pentagon is constructible!

So the Golden Ratio ϕ has already shown up! Regular readers of AtRiA will be familiar with this most famous ratio in mathematics. We refer them to the March 2013, March 2014, November 2016, March 2017, and July 2017 issues.

In Part II of this article we will return to the Golden Ratio for it will lead us to the Fibonacci sequence and more.

A nested sequence of regular pentagons

Extending all sides of the regular pentagon *ABCDE* we get a five-star figure *FHJGI* and then a pentagon *FGHIJ*.

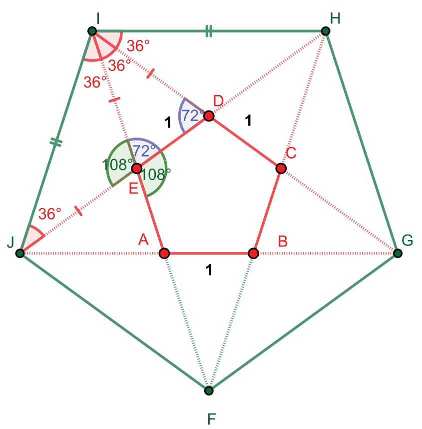

Figure 4

We first establish that the pentagon *FGHIJ* is a regular pentagon. This is actually not that hard. The first step is to realize that $\measuredangle DEI = 72°$ $= \measuredangle EDI$, because they are both the exterior angles of $\measuredangle AED$ and $\measuredangle CDE$ respectively. This in turn gives us that $|EI| = |DI|$. By the same argument we can show that $|EI| = |EJ|$ making $\triangle EIJ$ isosceles. Since $\measuredangle AED = \measuredangle JEI = 108°$, and $\triangle EIJ$ is isosceles we have $\measuredangle EIJ = 36°$. By a similar argument we can show that $\measuredangle DIH = 36°$. It is now easy to see that $\measuredangle JIH = 108°$. Further $\triangle JEI \cong \triangle DIH$ by SAS criterion and hence $JI = IH$. We can show similarly that all the sides and angles of the pentagon *FGHIJ* are equal making it a regular pentagon.

As you might have guessed we can continue this process and get a nested sequence of regular pentagons. The only thing to notice is the orientation is flipped in each iteration.

Since all regular pentagons are similar, the question is what is the scale factor for the sides and diagonals of each new pentagon?

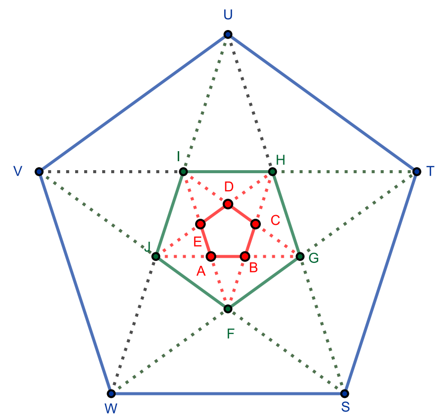

Figure 5

Let us look at Figure 4 again but now with the view of studying the lengths of the sides and diagonals of the pentagons.

From our previous work we know that $\measuredangle BAD = 72° = \measuredangle ABD$ and therefore from the ASA criterion we have $\triangle ABD \cong \triangle EDI$. Hence we see that $|EI| = |DI| = \phi$. By considering $\triangle ABD \cong \triangle CDH$, we get $|DH| = |HC| = \phi$.
Now consider the two triangles $\triangle DIH$ and $\triangle DEC$. They are similar because the corresponding angles are equal. We then have the following ratio:

$$\frac{|HI|}{\phi} = \frac{\phi}{1}$$

and hence $|HI| = \phi^2$. Hence the length of the sides of the new regular pentagon *FGHIJ* is ϕ^2. So it looks like each new regular pentagon is scaled by a factor of ϕ^2.

What about the diagonal of the regular pentagon *FGHIJ*? Since the diagonal of the regular pentagon *ABCDE* was ϕ the new diagonal should be ϕ^3. Let us see how we can prove this. There are two ways in which we can see this. The first is to recognize that

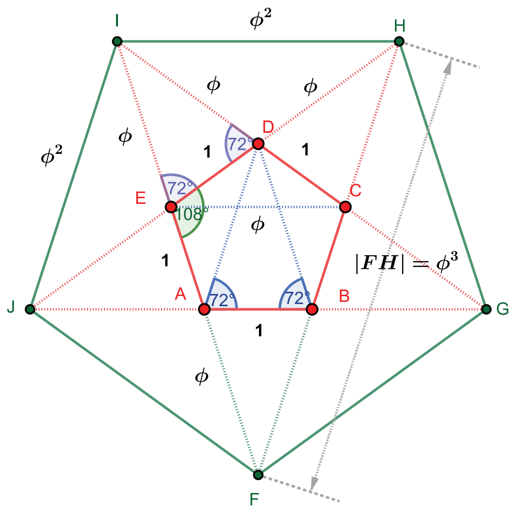

Figure 6

$\triangle IJF \sim \triangle DEC$ and obtain the ratio

$$\frac{|FI|}{\phi^2} = \frac{\phi}{1}$$

yielding $|FI| = \phi^3$. The other way is to use algebra! We know ϕ satisfies the equation $\phi^2 - \phi - 1 = 0$, rearranging we have $\phi^2 = \phi + 1$. Multiplying both sides by ϕ yields $\phi^3 = \phi^2 + \phi$; substituting ϕ^2 with $\phi + 1$ gives $\phi^3 = 2\phi + 1$. Now from $\triangle IHF$ we see $|FI| = |FH| = 2\phi + 1 = \phi^3$!

From the nested sequence of pentagons we have an infinite sequence of side, diagonal, side, diagonal, . . . :

$$1, \phi, \phi^2, \phi^3 \ldots.$$

In the next part of this article we will see how this sequence, along with a sister sequence, will lead to the famous Fibonacci sequence and as promised, we will see how they all come together in the icosahedron.

References

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SHASHIDHAR JAGADEESHAN has been teaching mathematics for over 30 years. He is a firm believer that mathematics is a human endeavour, and his interest lies in conveying the beauty of mathematics to students and demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He may be contacted at jshashidhar@gmail.com.